

Beyond the String Genus[☆]

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Abstract

In an earlier work we used a path integral analysis to propose a higher genus generalization of the elliptic genus. We found a cobordism invariant parametrized by Teichmüller space. Here we simplify the formula and study the behavior of our invariant under the action of the mapping class group of the Riemann surface. We find that our invariant is a modular function with multiplier just as in genus one.

1. Introduction

In reference [1] (BEG) we constructed a cobordism invariant using a supersymmetric sigma model with target space M , a string manifold, *i.e.*, $\frac{1}{2}p_1(M) = 0$. The case of genus one [2, 3, 4, 5] gives the string genus¹ à la [2, 3]. The cobordism invariant, the semiclassical approximation $Z_{sc}(M)$ of the partition function, was a function on odd spin Teichmüller space $\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$. Teichmüller space $\text{Teich}(\Sigma) = \text{Met}(\Sigma)/\text{Diff}_0(\Sigma)$ is the space of metrics of constant curvature² -1 on a Riemann surface Σ (genus $g > 1$) divided by the connected diffeomorphisms. Odd spin Teichmüller space $\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$ is the covering space of odd spin structures over the space of metrics, all divided by $\text{Diff}_0(\Sigma)$, a normal subgroup of $\text{Diff}(\Sigma)$.

The original purpose of the present paper was to refine our previous results so that our cobordism invariant would be a function over spin moduli space as opposed to spin Teichmüller space. We attempted to do so by dividing our previous result by the action of the mapping class group $\text{MCG}(\Sigma) = \text{Diff}(\Sigma)/\text{Diff}_0(\Sigma)$.

We remind the reader of the exact sequence,

$$1 \rightarrow \text{Torelli}(\Sigma) \rightarrow \text{MCG}(\Sigma) \rightarrow \text{Sp}(2g, \mathbb{Z}) \rightarrow 1, \quad (1.1)$$

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¹We use the string genus, *a.k.a* the Witten genus, instead of the elliptic genus used by topologists that corresponds to a more complicated field theory.

²The sectional curvature k is related to the Riemann tensor by $R_{abcd}^\Sigma = k(g_{ac}g_{bd} - g_{ad}g_{bc})$. The Ricci scalar is given by $R^\Sigma = 2k$. By curvature -1 we mean $k = -1$.

where $\text{Torelli}(\Sigma)$ is the normal subgroup of the mapping class group that is constant on $H_1(\Sigma, \mathbb{Z})$ and $\text{Sp}(2g, \mathbb{Z})$ is the symplectic group. This suggested that we first examine the action of $\text{Torelli}(\Sigma)$ on our invariant and afterwards the action of the symplectic group on what remains.

One of our main results is that Z_{sc} is a *modular* function on $\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$ with multiplier (Section 8.4).

We outline the contents of the paper.

In Section 2 we analyze the effect of the Torelli group on Z_{sc} ; we are left with the effect of $\text{Sp}(2g, \mathbb{Z}) = \text{MCG}(\Sigma) / \text{Torelli}(\Sigma)$ on Z_{sc} .

In Section 3 we generalize Ray-Singer torsion to the spinor case. We do so not only for its intrinsic interest but also because it has (7.24) as a corollary.

In Section 4 we focus on equation (4.2) shown below

$$Z_{\text{sc}} = \left(\frac{\text{vol } \Sigma}{\det_{1\perp} \Delta_0} \right)^n \left(\frac{\det' \partial_\delta}{N_\delta^2} \right)^n \int_M \prod_{k=1}^n \frac{\mathfrak{z}_\kappa(x_k z(\bar{h}_\delta^2))}{\vartheta[\kappa](x_k z(\bar{h}_\delta^2))}.$$

In the above $\det_{1\perp} \Delta_0$ is the determinant of the laplacian on the space of functions of the surface Σ orthogonal to the constants. An odd spin structure δ is a square root $\bar{K}_\delta^{1/2}$ of the anti-canonical line bundle \bar{K} of Σ . The operator ∂_δ maps $\Lambda^{0,1/2}(\Sigma)$ into $\Lambda^{1,1/2}(\Sigma)$ and is of index zero. Note that ∂_δ gives a family of elliptic operators parametrized by $\text{Met}(\Sigma)$ and ultimately parametrized by $\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$. It has generically a one dimensional kernel³ generated by \bar{h}_δ . Following Quillen [6] $\det' \partial_\delta$ is a section of the hermitian determinant line bundle of the ∂_δ family and $|N_\delta| = \|\bar{h}_\delta\|$. Riemann surface theory gives us an explicit expression for the square of the spinor:

$$h_\delta^2 = \sum_k \frac{\partial \vartheta[\delta](0)}{\partial z^k} \omega_k.$$

$(\omega_1, \dots, \omega_g)$ is a symplectic standard basis for the abelian differentials and $\vartheta[\delta](\cdot)$ is the Riemann theta function with characteristic δ .

We now describe the integral in Z_{sc} . κ is the vector of Riemann constants and $\vartheta[\kappa](0; \Omega) = 0$. The theta divisor Θ_κ near the origin of the Jacobian $J_0(\Sigma)$ is the zero set of \mathfrak{z}_κ , see Appendix B for the details. The x_k are the formal eigenvalues of the curvature 2-form on M whose symmetric polynomials express the Pontryagin classes of M . Thus the integral is a cobordism invariant depending on the metric of the Riemann surface.

In Section 5, we review the properties of $\text{Sp}(2g, \mathbb{Z})$ and compute the transformation properties of various quantities that appear in Z_{sc} . Using results in [7, 8] we find the transformation laws for the integral term in Z_{sc} and for h_δ^2 .

In Section 6 we give an exposition of quadratic refinements of cup product and its one-to-one correspondence with spin structures [9]. The action of the symplectic

³When $\dim \ker \partial_\delta > 1$, $\det' \partial_\delta = 0$ and therefore $Z_{\text{sc}} = 0$. The places where this occurs is a subvariety of spin Teichmuller space with complex codimension 1.

group on spin structures is derived by knowing its action on quarfs. We also introduce the subgroup $\Gamma_{1,2} \subset \mathrm{Sp}(2g, \mathbb{Z})$.

In Section 7 we show that the symplectic action lifts to an action on $\mathrm{Teich}_{\mathrm{odd}}^{1/2}(\Sigma)$, the covering space of odd spin structures. There is a preferred even spin structure $\hat{\mathcal{S}}$ and we find equation (7.34)

$$\frac{\det' \bar{\partial}_\delta}{N_\delta^2} = \frac{1}{\pi} \frac{\det \bar{\partial}(\hat{\mathcal{S}})}{\vartheta(0)},$$

where $\delta = \Omega a + b$ and Ω is the Riemann period matrix. The methods of this section give a generalization of the bosonization theorem to odd spin structures.

In Section 8 we study the determinant for the Dirac laplacian and we note that $f \in \mathrm{Diff}(\Sigma)$ (representing an element $\Lambda \in \mathrm{Sp}(2g, \mathbb{Z})$) induces an isometry between two f -related determinant line bundles. As a consequence, a phase factor $e^{i\xi(0,\Lambda)}$ appears in our computations. When $\Lambda \in \Gamma_{1,2}$ we can compute $e^{i\xi(0,\Lambda)}$. This section contains our main results, the simplest of which is that Z_{sc} is a modular function with multiplier when $\Lambda \in \Gamma_{1,2}$.

In Appendix A we discuss the conjugate linear isomorphism between $\Lambda^{1/2,0}(\Sigma)$ and $\Lambda^{1/2,1}(\Sigma)$.

In Appendix B we review some properties of the determinant line bundle for $\bar{\partial}$ -operators and discuss \mathfrak{z}_κ .

In Appendix C we relate our abstract modular transformation result to explicit formulas in genus one.

Finally we have a small nomenclature of recurring symbols.

Section 4 and Appendix B has some overlap with material in BEG. We rely heavily on the content of these sections in this paper and therefore include it here along with an expanded discussion of some topics in BEG.

1.1. Some Questions and Speculations

We note that when the Riemann surface degenerates our invariant factorizes. How does Z_{sc} fit with the modular geometry of Friedan and Shenker [10] that describes the behavior of string amplitudes when the Riemann surface degenerates?

Our main results give a genus, a map from a subring of the string cobordism ring to a subring of the functions on $\mathrm{Teich}_{\mathrm{odd}}^{1/2}(\Sigma)$. For $g = 1$ this leads to the string genus. Does our new genus give any new information about the string cobordism ring?

M.J. Hopkins suggests that we consider the Cayley plane, a string manifold of dimension 16, whose string genus vanishes. We would like to know if our genus Z_{sc} is non-zero for the Cayley plane; it might give us some information about string cobordism theory. This requires computing the function \mathfrak{z}_κ , an open problem which may be solvable for a hyperelliptic surface of genus 2.

2. The Action of $\mathrm{Diff}_0(\Sigma)$ and Torelli

2.1. The $\mathrm{Diff}_0(\Sigma)$ action

The action of $\mathrm{Diff}_0(\Sigma)$ on the partition function of a quantum field theory is well understood. The seminal work of Alvarez-Gaumé and Witten on gravitational

anomalies [11] initiated the subject. Gravitational anomalies are related to 1-cocycles in the group cohomology of $\text{Diff}_0(\Sigma)$. For simplicity we assume the partition function Z only depends on $\text{Met}(\Sigma)$, *i.e.*, $Z : \text{Met}(\Sigma) \rightarrow \mathbb{C}$. If $f \in \text{Diff}_0(\Sigma)$ and $\mathbf{g} \in \text{Met}(\Sigma)$ then the action on a metric is $f : \mathbf{g} \mapsto (f^{-1})^* \mathbf{g}$. The partition function behaves as $Z((f^{-1})^* \mathbf{g}) = \lambda(f, \mathbf{g}) Z(\mathbf{g})$ where $\lambda : \text{Diff}_0(\Sigma) \times \text{Met}(\Sigma) \rightarrow \mathbb{C}^\times$ is a 1-cocycle in the group cohomology of $\text{Diff}_0(\Sigma)$:

$$\lambda(f_1 \circ f_0, \mathbf{g}) = \lambda(f_1, (f_0^{-1})^* \mathbf{g}) \lambda(f_0, \mathbf{g}) \text{ where } f_0, f_1 \in \text{Diff}_0(\Sigma). \quad (2.1)$$

Because of the cocycle condition, the partition function may be interpreted as a section of a line bundle over $\text{Teich}(\Sigma) = \text{Met}(\Sigma) / \text{Diff}_0(\Sigma)$. Note that the discussion above is valid whether or not the metrics have constant curvature.

We make a brief but very important remark before we proceed with details in the ensuing subsections. Assume $h_0, h_1 \in \text{Diff}(\Sigma)$ represent the same element in the mapping class group then there exists $f \in \text{Diff}_0(\Sigma)$ such that $h_1 = f \circ h_0$. Next we observe that $Z((h_1^{-1})^* \mathbf{g}) = Z((f^{-1})^* \circ (h_0^{-1})^* \mathbf{g}) = \lambda(f, (h_0^{-1})^* \mathbf{g}) Z((h_0^{-1})^* \mathbf{g})$. Thus when we work in $\text{Teich}(\Sigma)$ and want to understand the action of the mapping class group on the *partition section* it does not matter which diffeomorphism we choose as a representative for an element in the mapping class group.

2.2. The Torelli Action

From the definition in the introduction of $\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$, the action of $\text{Torelli}(\Sigma)$ on $\text{Teich}(\Sigma)$ lifts to an action of $\text{Torelli}(\Sigma)$ on $\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$ that we will denote by $\text{Torelli}_{\text{odd}}^{1/2}(\Sigma)$. We now discuss the action of $\text{Torelli}_{\text{odd}}^{1/2}(\Sigma)$ on the determinant line bundle \mathcal{L} of [BEG, Theorem 9.1], where $\mathcal{L} = ((\text{DET}(\bar{\partial}_\delta))^*)^n$, $\dim M = 2n$, $\bar{\partial}_\delta : \Lambda^{1/2,0}(\Sigma) \rightarrow \Lambda^{1/2,1}(\Sigma)$, and $\Lambda^{1/2,0}(\Sigma) = \sqrt{K}^\delta$ is the square root of the canonical bundle corresponding the odd spin structure δ .

Lemma 2.1. $\text{Torelli}_{\text{odd}}^{1/2}(\Sigma)$ acting on $\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$ leaves \mathcal{L} invariant.

We study $\bar{\partial}_\delta$ in the generic case where it has a one dimensional kernel. The determinant line bundle $\text{DET}(\bar{\partial}_\delta)$ is isomorphic to the dual lie bundle of a line subbundle in $H^{1,0}(\Sigma) \subset \Lambda^{1,0}(\Sigma)$. Observe that $\Lambda^{1/2,1}(\Sigma)$ is the linear algebraic dual space of $\Lambda^{1/2,0}(\Sigma)$ by using wedge product and integration over Σ ; we also use the standard inner product on Σ to get an inner product on $\Lambda^{1/2,0}(\Sigma)$. An elementary computation shows that if h_δ is in $\ker \bar{\partial}_\delta$ then its linear algebraic dual using the inner product is in $\ker \bar{\partial}_\delta^*$. Thus the determinant line bundle is the dual line bundle of the line in $H^{1,0}(\Sigma)$ determined by h_δ^2 . Moreover, $\text{Torelli}_{\text{odd}}^{1/2}(\Sigma)$ acting on $\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$ leaves the determinant line bundle $\text{DET}(\bar{\partial}_\delta)$ fixed because it sends the one dimensional kernel of $\bar{\partial}_\delta$ to the one dimensional kernel of the transformed $\bar{\partial}_\delta$.

If $f \in \text{Diff}(\Sigma)$ then f induces a transformation in homology $f_* : H_1(\Sigma, \mathbb{Z}) \rightarrow H_1(\Sigma, \mathbb{Z})$. Define a normal subgroup $\widetilde{\text{Torelli}}(\Sigma) \triangleleft \text{Diff}(\Sigma)$ by

$$\widetilde{\text{Torelli}}(\Sigma) = \{f \in \text{Diff}(\Sigma) \mid f_* = \text{Id}\}.$$

It is a normal subgroup because if $g \in \text{Diff}(\Sigma)$ then $(gf g^{-1})_* = g_* f_* g_*^{-1} = g_* \text{Id } g_*^{-1} = \text{Id}$. Note that $\text{Torelli}(\Sigma) = \widetilde{\text{Torelli}(\Sigma) / \text{Diff}_0(\Sigma)}$ and that $\text{Sp}(2g, \mathbb{Z}) = \text{Diff}(\Sigma) / \widetilde{\text{Torelli}(\Sigma)}$.

Observe that once a standard symplectic basis $(\mathbf{a}_i, \mathbf{b}_j)$ of cycles for $H_1(\Sigma, \mathbb{Z})$ is chosen then the abelian differentials ω_i are uniquely determined and depend only on the homology classes $([\mathbf{a}_i], [\mathbf{b}_j])$.

If $f \in \text{Diff}(\Sigma)$, let $f : (\Sigma, g) \rightarrow (\Sigma, \tilde{g})$ where g and \tilde{g} are the respective metrics. Let $c_\alpha = (\mathbf{a}_i, \mathbf{b}_j)$ be a choice for the standard cycles on (Σ, g) . A mapping f is of Torelli type if it preserves the homology. For such an f there is an induced transformation on cycles z that gives $f_* z = z + \partial \tilde{q}_\alpha$. Hence the Riemann period matrix is invariant:

$$\tilde{\Omega}_{ij} = \int_{\tilde{\mathbf{b}}_i} \tilde{\omega}_j = \int_{f_* \mathbf{b}_i} \tilde{\omega}_j = \int_{\mathbf{b}_i} f^* \tilde{\omega}_j = \Omega_{ij},$$

and moreover $\delta_{ij} = \int_{\mathbf{a}_i} \omega_j = \int_{\tilde{\mathbf{a}}_i} \tilde{\omega}_j$.

3. Ray-Singer Torsion Revisited

Fix a fiducial metric $\hat{g} \in \text{Met}_{\text{all}}(\Sigma)$, the space of all metrics on Σ . The metric \hat{g} determines a complex structure within the $3(g-1)$ complex dimensional space of complex structures. We restrict our discussion to surfaces with genus $g > 1$.

For Riemann surfaces, the complex Ray-Singer torsion theorem [12, Theorem 2.1] is a consequence of the conformal anomaly. Let F_χ be the flat holomorphic line bundle associated with the character $\chi : \pi_1(\Sigma) \rightarrow S^1$. F_χ comes equipped with a hermitian metric that depends only on the complex structure. The sections of $K^n \otimes \bar{K}^m$ are the “ (n, m) -forms” and are denoted by $T_{n,m}$. The hermitian metric on Σ allows us to identify (n, m) -forms with $(n-m, 0)$ -forms. Let $\bar{\partial}_n : T_{n,0} \otimes F_\chi \rightarrow T_{n,1} \otimes F_\chi$ be the basic operator, $\bar{\partial}_n^* : T_{n,1} \otimes F_\chi \rightarrow T_{n,0} \otimes F_\chi$ be its hermitian adjoint and let $\Delta_n^{(-)} = 2\bar{\partial}_n^* \bar{\partial}_n$ be the corresponding laplacian. Let $\{\phi_a\}$ be a basis for $\ker \bar{\partial}_n$, the holomorphic sections of $T_{n,0} \otimes F_\chi$. The basis can be chosen to be independent of conformally scaling the fiducial metric by $e^{2\sigma}$, *i.e.*, it only depends on the complex structure. Because of the hermitian metrics on K and F_χ , $\ker \bar{\partial}_n^* \subset T_{n,1} \otimes F_\chi \approx T_{n-1,0} \otimes F_\chi$ and may be identified with the holomorphic sections of $\bar{\partial}_{1-n} : T_{1-n,0} \otimes F_\chi^{-1} \rightarrow T_{1-n,1} \otimes F_\chi^{-1}$. We also have the dual space identification

$$T_{1-n,0} \otimes F_\chi^{-1} \approx (T_{n-1,0} \otimes F_\chi)^* . \quad (3.1)$$

Let $\{\psi_\alpha\}$ be the holomorphic sections of $T_{1-n,0} \otimes F_\chi^{-1}$ (chosen to be independent of the conformal factor σ). The conformal anomaly implies that under an infinitesimal conformal change of the metric $\hat{g} \rightarrow e^{2(\delta\sigma)} \hat{g}$ we have [13]

$$\delta_\sigma \log \left(\frac{\det' \Delta_n^{(-)}}{\det \langle \psi_\alpha, \psi_\beta \rangle \det \langle \phi_a, \phi_b \rangle} \right)_\chi = -\frac{1+6n(n-1)}{6\pi} \int_\Sigma d^2 z \sqrt{\hat{g}} \hat{R} (\delta\sigma) . \quad (3.2)$$

The term inside the parentheses on the left hand side of the equation is the Quillen metric of the determinant line bundle $\text{DET}(\bar{\partial}_n)$. The determinant term associated

with $\ker \bar{\partial}_n^*$ appears in the denominator because of the dual space identification given in eq. (3.1). This formula is valid for $2n \in \mathbb{Z}$. Note that the right hand side is independent of F_χ .

The Ray-Singer torsion results correspond⁴ to the case $n = 0$. Let χ and χ' be two non-trivial characters. For both characters, $\ker \bar{\partial}_0 = \{0\}$ and $\dim \ker \bar{\partial}_0^* = g - 1$. Also the metric on $T_{1,0} \otimes F_\chi^{-1}$ is independent of the conformal factor and therefore the term $\det \langle \psi_\alpha, \psi_\beta \rangle$ in the left hand side of (3.2) does not change under a conformal variation. Putting all this information together gives

$$\delta_\sigma \left(\log \det \Delta_0^{(-)}(\chi) - \log \Delta_0^{(-)}(\chi') \right) = 0. \quad (3.3)$$

This is the Ray-Singer result for complex analytic torsion on Riemann surfaces. It says that the ratio $(\det \Delta_0^{(-)}(\chi)) / (\det \Delta_0^{(-)}(\chi'))$ only depends on the complex structure.

An immediate consequence of (3.2) is

Theorem 3.1 (Generalized Ray-Singer Torsion on Riemann Surfaces). *Consider a collection $\{(n_r, \chi_r, k_r)\}_{r=1}^N$ where $2n_r \in \mathbb{Z}$, $\chi_r : \pi_1(\Sigma) \rightarrow S^1$ is a character, and $k_r \in \mathbb{Z}$. If this collection satisfies*

$$\sum_{r=1}^N k_r [1 + 6n_r(n_r - 1)] = 0$$

then

$$\sum_{r=1}^N k_r \log \left(\frac{\det' \Delta^{(-)}}{\det \langle \psi_\alpha, \psi_\beta \rangle \det \langle \phi_a, \phi_b \rangle} \right)_{n_r, \chi_r}$$

only depends on the complex structure and is independent of the choice of hermitian metric on Σ .

The two best known examples of this theorem in string theory are the 26 dimensional bosonic string [14] with $\{(0, 1, 26/2), (-1, 1, -1)\}$, and the 10 dimensional superstring [15] with collection

$$\{(0, 1, 10/2), (1/2, \chi, -10/2), (-1, 1, -1), (-1/2, \chi^{-1}, 1)\}$$

associated with 10 bosons, 10 Majorana fermions, diffeomorphisms (vector fields), super-diffeomorphisms (square root of vector fields). In the above χ can be any character corresponding to a spin structure.

3.1. Ray-Singer Torsion for Spinors

We can be very explicit in general genus in the case of spinors. Pick a reference point $P_0 \in \Sigma$ and in the standard fashion identify $J_0(\Sigma)$ with $J_{g-1}(\Sigma)$ as discussed in Appendix B. The character χ corresponds to a point $u \in J_0(\Sigma)$. There is a special spin structure $\hat{\mathcal{S}}$ such that the determinant of the laplacian acting on $\hat{\mathcal{S}} \otimes F_\chi$ is given

⁴The $T_0 = T_1$ result of Ray and Singer is related to the two laplacians we can define.

by (7.20) where $u \in J_0(\Sigma)$ is the point corresponding to F_χ . This example shows that $(\det \Delta(u))/(\det \Delta(u'))$ only depends on the complex structure in agreement with the generalized Ray-Singer theorem.

We introduce the notation

$$Q(u) = \frac{\det' D(u)^* D(u)}{\det \langle \psi_\alpha, \psi_\beta \rangle \det \langle \phi_a, \phi_b \rangle} \quad (3.4)$$

for convenience and define subvarieties $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \dots$ of $J_0(\Sigma)$ where \mathcal{V}_k is the set of points $u \in J_0(\Sigma)$ where $\dim \ker D(u) = k$. Note that the theta divisor is given by $\Theta = \cup_{k=1}^\infty \mathcal{V}_k$. Because the Dirac operator has index zero, the matrices $\langle \psi_\alpha, \psi_\beta \rangle$ and $\langle \phi_a, \phi_b \rangle$ are the same size. We have seen that if $u, u' \in \mathcal{V}_0$ then $Q(u)/Q(u')$ is independent of the choice of hermitian metric on Σ . Similarly, if $u \in \mathcal{V}_k$ and $v \in \mathcal{V}_l$ then $Q(u)/Q(v)$ will be independent of the choice of hermitian metric on Σ .

In general, ϕ_a is a holomorphic section of $\hat{\mathcal{S}} \otimes F_\chi$ and ψ_α is a holomorphic section of $\hat{\mathcal{S}} \otimes F_\chi^{-1}$. When F_χ corresponds to a semi-characteristic so that $\hat{\mathcal{S}} \otimes F_\chi$ is a spin structure, F_χ^2 is the trivial bundle. Thus $F_\chi \approx F_\chi^{-1}$ and we can identify the determinants in the denominator of (3.4). In the generic case where all the odd spin structure are in \mathcal{V}_1 we have the explicit computations (7.23) (an example involving \mathcal{V}_0 and \mathcal{V}_1), and (7.24) (an example only involving \mathcal{V}_1).

4. The Bosonic Determinant

We rewrite our key formula [BEG, (4.12)] by changing the orientation of the surface $z \rightarrow \bar{z}$. Now the semiclassical partition “function” (section) becomes

$$Z_{\text{sc}} = (\text{vol } \Sigma)^n \left(\frac{\det' \partial_\delta}{\bar{N}_\delta^2} \right)^n \int_M \det [D(\partial_0 \otimes I_{2n})]^{-1/2}, \quad (4.1)$$

here $\dim M = 2n$, $D = *i(\bar{\partial} \otimes I_{2n} + \mathbf{A}^{0,1})$ acting on $\Lambda^{1,0}(\Sigma) \otimes X^*(TM)$ with $X : \Sigma \rightarrow M$ a constant map. We integrate over the space of constant maps M . We assume the odd spin structure δ is generic, so $\ker \partial_\delta$ is 1-dimensional and chose a non-zero element in the kernel $\bar{h}_\delta \in \Lambda^{0,1/2}(\Sigma)$ and also $\bar{N}_\delta \in \mathbb{C}$ such that $\bar{h}_\delta/\bar{N}_\delta$ has norm 1. Hence $\bar{h}_\delta/\bar{N}_\delta$ is an element of norm 1 in $\Lambda^{0,1/2}(\Sigma)$. The term $\mathbf{A}^{0,1}$ in the definition of D is $\bar{h}_\delta^2 \otimes \mathcal{R}/2\pi$ with \mathcal{R} the curvature 2-form of M pulled back via the constant map X .

Z_{sc} depends on a metric \mathbf{g} on Σ and an odd spin structure δ . In the Section 6 we review how a choice of symplectic basis b for $H_1(\Sigma, \mathbb{Z})$ fixes an even spin structure \sqrt{K}^b . Adding an appropriate element $w \in H^1(\Sigma, \mathbb{Z}_2)$ gives an odd spin structure so $Z_{\text{sc}} = Z_{\text{sc}}(\mathbf{g}, b, w)$. If $f \in \text{Diff}(\Sigma)$ then f induces a map $(\mathbf{g}; b, w) \mapsto ((f^{-1})^* \mathbf{g}; f_* b, (f^{-1})^* w)$.

In BEG we showed that

$$Z_{\text{sc}} = \left(\frac{\text{vol } \Sigma}{\det_{1^\perp} \Delta_0} \right)^n \left(\frac{\det' \partial_\delta}{\bar{N}_\delta^2} \right)^n \int_M \prod_{k=1}^n \frac{\mathfrak{z}_\kappa(x_k z(\bar{h}_\delta^2))}{\vartheta[\kappa](x_k z(\bar{h}_\delta^2))}. \quad (4.2)$$

See the Introduction for the definitions of the terms except for \mathfrak{z}_κ that can be found in Appendix B. Riemann surface theory [8] gives us an explicit expression for the

square of the spinor:

$$h_\delta^2 = \sum_k \frac{\partial \vartheta[\delta](0)}{\partial z^k} \omega_k. \quad (4.3)$$

Formula (4.1) contains a specific 1-form with curvature zero. The flatness arises because the 1-form is the product of the pullback of the curvature on the target space M by the constant map, and the square of the anti-holomorphic spinor.

It is useful to consider the family of operators $D = i * (\bar{\partial} + A^{0,1})$ with $A^{0,1}$ a flat connection. Here $D : \Lambda^{1,0}(\Sigma) \rightarrow \Lambda^{0,0}(\Sigma)$ is parametrized by $J_0(\Sigma)$. The determinant line bundle $\mathcal{L} = \text{DET } D \rightarrow J_0(\Sigma)$ has a Quillen hermitian metric with connection ν and curvature $d\nu$ given by the standard translationally invariant polarization form on $J_0(\Sigma)$. It also has a unique holomorphic cross section (up to scale), see [BEG, Appendix C].

The computation of the bosonic determinant in (4.1) involves three steps.

1. In Section 4.1 we will lift \mathcal{L} and its holomorphic cross section to the covering space $H^{0,1}(\Sigma)$ of $J_0(\Sigma)$, then trivialize the lift hence making the cross section a function which we will identify as a ϑ -function.
2. In Appendix B, we use elliptic analysis to study \mathcal{L} and its unique holomorphic section.
3. Combining the two previous items leads to a formula for the aforementioned determinant after studying a PDE as discussed in [BEG, Section 6].

4.1. Trivializing the Determinant Line Bundle

Remember⁵ that the jacobian is defined by $J_0(\Sigma) = H^{0,1}(\Sigma)/L_\Omega$. We identify $H^{0,1}(\Sigma)$ with \mathbb{C}^g by choosing a basis of $H^{0,1}(\Sigma)$ given by formula (4.4) with z_j the coordinates of \mathbb{C}^g . With this convention the lattice $L_\Omega \subset H^{0,1}(\Sigma)$ is given by (4.5).

$$A^{0,1} = 2\pi i \sum z_j (\Omega - \bar{\Omega})_{jk}^{-1} \bar{\omega}_k, \quad (4.4)$$

$$B_{nm}^{0,1} = 2\pi i \sum (m + \Omega n)_j (\Omega - \bar{\Omega})_{jk}^{-1} \bar{\omega}_k, \text{ where } m, n \in \mathbb{Z}^g. \quad (4.5)$$

With these conventions, the quasiperiodicity of the theta function are associated with $z \rightarrow z + m + \Omega n$.

The jacobian torus defined above is the dual torus to the one normally used by algebraic geometers. If (α_j, β_k) is a symplectic basis for $H^1(\Sigma, \mathbb{R})$ in terms of harmonic 1-forms then the abelian differentials are given by $\omega_j = \alpha_j + \sum_k \Omega_{jk} \beta_k$. The algebro-geometric jacobian is $H^{1,0}(\Sigma)$ modulo the integer lattice $H^1(\Sigma, \mathbb{Z})$, see Appendix B. We know that the linear algebraic dual of $H^{0,1}(\Sigma)$ is $H^{1,0}(\Sigma)$. There is a hermitian inner product on $H^{1,0}(\Sigma)$ and therefore there is a conjugate linear isomorphism with $H^{0,1}(\Sigma)$ that identifies the basis vectors ω_i with $\sum_k (\Omega - \bar{\Omega})_{jk} \bar{\omega}_k$ up to an overall normalization.

⁵The content in this section is required for the flow of this paper; there is overlap with work in BEG.

Let $\pi : H^{0,1}(\Sigma) \rightarrow J_0(\Sigma)$ be the standard projection and define the pull back line bundle $\tilde{\mathcal{L}} = \pi^* \mathcal{L} \rightarrow H^{0,1}(\Sigma)$ with connection $\tilde{\nu} = \pi^* \nu$. On $H^{0,1}(\Sigma)$ the 1-form

$$\rho = \frac{i}{2\pi} \int_{\Sigma} \overline{A^{0,1}} \wedge d_{H^{0,1}(\Sigma)} A^{0,1} \quad (4.6)$$

has the property that $0 = d(\tilde{\nu} - \rho) = \bar{\partial}(\tilde{\nu} - \rho)$. In [BEG, Section 6] we used the flat connection $\tilde{\nu} - \rho$ to trivialize $\tilde{\mathcal{L}}$. We briefly review a slight modification of that discussion here. Let $\tilde{\mathcal{L}}_0$ be the fiber over $0 \in H^{0,1}(\Sigma)$. To identify the fiber $\tilde{\mathcal{L}}_0$ with \mathbb{C} we choose an arbitrary non-zero point $\hat{\sigma}_0 \in \tilde{\mathcal{L}}_0$. Given two points $A_0, A_1 \in H^{0,1}(\Sigma)$, an integral $\int_{A_0}^{A_1} \dots$ is always taken along the straight segment⁶ joining the two points. We defined the *flat trivialization* $\varphi : \tilde{\mathcal{L}} \rightarrow H^{0,1}(\Sigma) \times \mathbb{C}$ by using the flat connection $\tilde{\nu} - \rho$ to give us a holomorphic trivialization. More explicitly, let σ be a point in the fiber of $\tilde{\mathcal{L}}$ over A then parallel transport σ along the straight segment from A to 0 to obtain a point $\sigma_0 \in \tilde{\mathcal{L}}_0$. The trivialization map is $\varphi : \sigma \mapsto (A, \sigma_0/\hat{\sigma}_0)$. Abusing notation slightly, we write

$$\sigma(A) = (\varphi\sigma)(A) \exp\left(-\int_0^A (\tilde{\nu} - \rho)\right) \hat{\sigma}_0. \quad (4.7)$$

To make things more standard we defined a slightly different trivialization that we called the *standard trivialization* Φ by multiplying the above by a non-vanishing holomorphic function on \mathbb{C}^g . The standard trivialization is defined by

$$(\Phi\sigma)(A) = \exp\left(-\pi i \sum_{j,k} z_j (\Omega - \bar{\Omega})_{jk}^{-1} z_k\right) (\varphi\sigma)(A). \quad (4.8)$$

If s is any section of \mathcal{L} and if $\tilde{s} = \pi^* s$ is the pull back section to $\tilde{\mathcal{L}}$ then $\tilde{s}(A+B) = \tilde{s}(A)$ for any lattice vector B . Consequently the pull back of any section of \mathcal{L} in the trivialized bundle (trivialized by the standard trivialization) is represented by a function $\Phi\tilde{s}$ with quasi-periodicity properties

$$(\Phi\tilde{s})(A+B) = \chi(B_{nm}) \times e^{-\pi i \sum_{j,k} n_j \Omega_{jk} n_k} e^{-2\pi i \sum n_j z_j} (\Phi\tilde{s})(A). \quad (4.9)$$

The above is the standard transformation law for a theta function with lattice character

$$\chi(B_{nm}) = e^{-\pi i \sum m_j n_j} e^{\int_0^B \nu} \quad (4.10)$$

$$= e^{-\pi i \sum m_j n_j} \text{hol}(\gamma_{nm}^0)^{-1}, \quad (4.11)$$

where $\text{hol}(\gamma_{nm}^0)$ is the holonomy of the Quillen connection. The closed curve γ_{nm}^0 is the projection into the jacobian of the straight segment from 0 to B_{nm} in $H^{0,1}(\Sigma)$.

⁶Many of our integrals involve a flat connection so the choice of integration path is irrelevant. It will matter to us when we project the curve down to the $J_0(\Sigma)$ and try to interpret results geometrically.

The bundle $\mathcal{L} \rightarrow J_0(\Sigma)$ has a unique holomorphic cross section (up to scale). If we write

$$\chi(B_{nm}) = e^{-2\pi i n \cdot b} e^{2\pi i m \cdot a} \quad (4.12)$$

then (4.9) is the transformation law for the function $\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right](z)$. In Appendix B we show that the characteristic $\left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]$ associated with the holomorphic section of $\mathcal{L} = \text{DET}(D) \rightarrow J_0(\Sigma)$ is $\kappa = \Omega a + b$ where κ is the vector of Riemann constants.

The line bundle $\mathcal{L} = \text{DET}(D) \rightarrow J_0(\Sigma)$ has a unique holomorphic section θ_κ up to scale. The results above state that the pullback section $\pi^* \theta_\kappa$ on $\tilde{\mathcal{L}} = \pi^* \mathcal{L}$ is related to the ϑ -function on the trivialized bundle $H^{0,1}(\Sigma) \times \mathbb{C}$ by

$$(\pi^* \theta_\kappa)(A) = \exp \left(+\pi i \sum_{j,k} z_j (\Omega - \bar{\Omega})_{jk}^{-1} z_k \right) \vartheta[\kappa](z) \exp \left(- \int_0^A (\tilde{\nu} - \rho) \right) \tilde{\sigma}_0. \quad (4.13)$$

It is explicit from the above that the lift of the divisor Θ_κ of θ_κ to $H^{0,1}(\Sigma)$ is the same as the zero set of $\vartheta[\kappa](\cdot)$.

5. Symplectic Action

5.1. Facts About the Symplectic Group

Let $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{F})$ for some field \mathbb{F} . Then

$$\Lambda^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \Lambda = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (5.1)$$

so that

$$D^t A - B^t C = I, \quad A^t C - C^t A = 0, \quad B^t D - D^t B = 0.$$

Thus $A^t C$ and $B^t D$ are symmetric matrices and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \quad (5.2)$$

is a left inverse hence

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} = \begin{pmatrix} AD^t - BC^t & -AB^t + BA^t \\ CD^t - DC^t & -CB^t + DA^t \end{pmatrix},$$

implying

$$D^t A - B^t C = I, \quad A^t C - C^t A = 0, \quad B^t D - D^t B = 0, \quad (5.3a)$$

$$AD^t - BC^t = I, \quad AB^t - BA^t = 0, \quad CD^t - DC^t = 0. \quad (5.3b)$$

Equations (5.3) completely characterize a symplectic matrix.

Equation (5.3a) implies (5.3b): multiply the first of (5.3) on the left by A

$$\begin{aligned} 0 &= -A + AD^t A - AB^t C = (-I + AD^t - BC^t)A + B \underbrace{C^t A}_{A^t C} - AB^t C \\ &= (-I + AD^t - BC^t)A + (BA^t - AB^t)C. \end{aligned}$$

Multiply the first of (5.3) on the right by D^t

$$\begin{aligned} 0 &= -D^t + D^t A D^t - B^t C D^t = D^t(-I + A D^t - B C^t) + \underbrace{D^t B}_{B^t D} C^t - B^t C D^t \\ &= D^t(-I + A D^t - B C^t) + B^t(D C^t - C D^t). \end{aligned}$$

These equations may be written in matrix form (after transposing the second group) as

$$\begin{pmatrix} -I + A D^t - B C^t & B A^t - A B^t \\ C D^t - D C^t & -I + D A^t - C B^t \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0.$$

Since the matrix on the right is invertible the one on left must be the zero matrix giving (5.3b).

5.2. Transformation Laws

Pick a symplectic basis $(\mathbf{a}_j, \mathbf{b}_k)$ for $H_1(\Sigma, \mathbb{Z})$ and represent the dual basis via harmonic differentials (α_j, β_k) . The standard normalized holomorphic abelian differentials are given by $\omega_j = \alpha_j + \sum_k \Omega_{jk} \beta_k$ where Ω is the period matrix. The action of $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$ gives new bases⁷ $(\mathbf{a}', \mathbf{b}')$ and (α', β') :

$$\begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \quad \text{and} \quad (\alpha \ \beta) = (\alpha' \ \beta') \begin{pmatrix} D & C \\ B & A \end{pmatrix}. \quad (5.4)$$

Notice that

$$\begin{pmatrix} D & C \\ B & A \end{pmatrix}^{-1} = \begin{pmatrix} A^t & -C^t \\ -B^t & D^t \end{pmatrix}. \quad (5.5)$$

Under the symplectic action

$$\omega = \omega'(C\Omega + D) \quad \text{and} \quad \Omega' = (A\Omega + B)(C\Omega + D)^{-1}. \quad (5.6)$$

where ω is the row vector $(\omega_1, \dots, \omega_g)$. The covering space of the jacobian torus $J_0(\Sigma)$ is \mathbb{C}^g . In terms of standard coordinates on \mathbb{C}^g , the jacobian torus is given by the identifications $z \sim z + m + n\Omega$ where we write the coordinates as a row vector. Under the action of Λ , the transformed torus is described by $z' \sim z' + m' + n'\Omega'$ so that

$$z = z'(C\Omega + D). \quad (5.7)$$

Note the $\text{Sp}(2g, \mathbb{Z})$ invariance:

$$\sum_j \omega_j \frac{\partial}{\partial z_j} = \sum_j \omega'_j \frac{\partial}{\partial z'_j}. \quad (5.8)$$

According to Fay [7, 16] a theta function with generic characteristics transforms under symplectic transformations by

$$\vartheta \left[\begin{smallmatrix} a' \\ b' \end{smallmatrix} \right] (z'; \Omega') = \varepsilon(\Lambda) e^{-i\pi\phi(a,b,\Lambda)} e^{\sum Q_{ij} z_i z_j} \det(C\Omega + D)^{1/2} \vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z; \Omega), \quad (5.9)$$

⁷We use the notation that a primed quantity is the symplectic transform of the unprimed quantity.

where $\varepsilon : \text{Sp}(2g, \mathbb{Z}) \rightarrow \mathbb{Z}_8$ is a phase independent of z and Ω ,

$$\phi(a, b, \Lambda) = aD^t Ba + bC^t Ab - [2aB^t Cb + (aD^t - bC^t)(AB^t)_d], \quad (5.10)$$

and

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \frac{1}{2} \begin{pmatrix} (CD^t)_d \\ (AB^t)_d \end{pmatrix}. \quad (5.11)$$

In the above $(AB^t)_d$ means the diagonal entries of the matrix product as a column vector. We do not need an explicit form for the symmetric matrix Q_{ij} because the condition $\frac{1}{2} p_1(M) = 0$ eliminates that term in our computations. Later we show that transformation law (5.11) is equivalent to the transformation law for quarks (6.5).

Let δ be the odd theta characteristic for a spin structure with holomorphic spinor h_δ . As noted in the introduction

$$h_\delta^2 = \sum_k \frac{\partial \vartheta[\delta]}{\partial z_k}(0; \Omega) \omega_k. \quad (5.12)$$

$\vartheta[\delta](0; \Omega) = 0$, (5.9) and (5.8) implies

$$(h'_{\delta'})^2 = \varepsilon(\Lambda) e^{-i\pi\phi(\delta, \Lambda)} \det(C\Omega + D)^{1/2} h_\delta^2. \quad (5.13)$$

Using

$$\int_\Sigma \bar{\omega}_k \wedge \omega_j = (\Omega - \bar{\Omega})_{kj}$$

and (4.4) implies that if $A^{0,1}$ is a 1-form in $H^{0,1}(\Sigma)$ then

$$2\pi i z_k(A^{0,1}) = \int_\Sigma A^{0,1} \wedge \omega_k. \quad (5.14)$$

If $H(\Omega)$ is the Hodge matrix associated with period matrix Ω , then

$$|\det(C\Omega + D)|^2 \det H(\Omega') = \det H(\Omega). \quad (5.15)$$

In our applications the object that enters is not the holomorphic section of \sqrt{K}^δ but the anti-holomorphic section of $\sqrt{\bar{K}}^\delta$ which we denote by \bar{h}_δ . The standard coordinate for $\bar{h}_\delta^2 \in H^{0,1}(\Sigma)$ is

$$2\pi i z_k(\bar{h}_\delta^2) = \int_\Sigma \bar{h}_\delta^2 \wedge \omega_k = \sum_j \overline{\frac{\partial \vartheta[\kappa]}{\partial z_j}(0; \Omega)} (\Omega - \bar{\Omega})_{jk}. \quad (5.16)$$

Using the complex conjugate of (5.13) gives

$$\begin{aligned} 2\pi i z'((\bar{h}'_{\delta'})^2)(C\Omega + D) &= \left(\int_\Sigma (\bar{h}'_{\delta'})^2 \wedge \omega' \right) (C\Omega + D), \\ &= \overline{\varepsilon(\Lambda)} e^{+i\pi\phi(\delta)} \det(C\bar{\Omega} + D)^{1/2} \int_\Sigma (\bar{h}_\delta)^2 \wedge \omega, \\ &= \overline{\varepsilon(\Lambda)} e^{+i\pi\phi(\delta)} \det(C\bar{\Omega} + D)^{1/2} 2\pi i z(\bar{h}_\delta^2). \end{aligned}$$

Thus the standard coordinates for $(\bar{h}'_{\delta'})^2$ and the standard coordinates for \bar{h}_δ^2 are related by

$$z'((\bar{h}'_{\delta'})^2)(C\bar{\Omega} + D) = \overline{\varepsilon(\Lambda)} e^{+i\pi\phi(\delta)} \det(C\bar{\Omega} + D)^{1/2} z(\bar{h}_\delta^2). \quad (5.17)$$

This differs by a scale from the standard relationship (5.7) for the coordinates between corresponding points in $H^{0,1}(\Sigma)$ under the action of $\mathrm{Sp}(2g, \mathbb{Z})$.

To work out the transformation properties of $\mathfrak{z}_\kappa/\vartheta[\kappa](\cdot)$ we use: (1) the relation (B.4), (2) Remark Appendix B.3, (3) M is a string manifold, (4) equation (4.2) has a factor that involves \int_M . Putting all these observations together gives

Theorem 5.1. *Let M be a string manifold with $\dim M = 2n$, if $\Lambda \in \mathrm{Sp}(2g, \mathbb{Z})$ is represented by $f \in \mathrm{Diff}(\Sigma)$, then under the action of f we have*

$$\begin{aligned} \int_M \prod_{r=1}^n \frac{\mathfrak{z}_{\kappa'}(x_r z'((\bar{h}'_{\delta'})^2); \Omega')}{\vartheta[\kappa'](x_r z'((\bar{h}'_{\delta'})^2); \Omega')} &= \varepsilon(\Lambda)^{-n} e^{+i\pi n\phi(\delta)} \det(C\bar{\Omega} + D)^{n/2} \\ &\times \int_M \prod_{r=1}^n \frac{\mathfrak{z}_\kappa(x_r z(\bar{h}_\delta^2); \Omega)}{\vartheta[\kappa](x_r z(\bar{h}_\delta^2); \Omega)}. \end{aligned} \quad (5.18)$$

For a generic odd spin structure δ , the determinant line bundle $\mathrm{DET}(\partial_\delta)$ is the line bundle dual to the complex line bundle generated by \bar{h}_δ^2 in $H^{0,1}(\Sigma)$. Equation (5.13) tells us how the line transforms under symplectic transformations thus we see that (5.18) is the transformation law for a section of the determinant line bundle $\mathrm{DET}(\partial_\delta)^n$ in the trivialization given by $(1/\bar{h}_\delta^2)^n$ hence

Theorem 5.2. *The function*

$$\int_M \prod_{r=1}^n \frac{\mathfrak{z}_\kappa(x_r z(\bar{h}_\delta^2); \Omega)}{\vartheta[\kappa](x_r z(\bar{h}_\delta^2); \Omega)} : \mathrm{Teich}_{odd}^{1/2}(\Sigma) \rightarrow \mathbb{C} \quad (5.19)$$

represents a section of the line bundle $\mathrm{DET}(\partial_\delta)^n \rightarrow \mathcal{M}_{odd}^{1/2}(\Sigma)$.

A drawback of the two theorems above is that the integrals are not holomorphic. They will be reformulated shortly to behave in a more holomorphic form analogous to the genus 1 case. To do this we have to understand the transformation of fermion determinants under the geometric symplectic action. Much of our intuition draws from the genus 1 case and Appendix C is devoted to it and the insights it gives for higher genus. We hope this review of the genus one case will be illuminating.

6. Quadratic Refinements and Spin Structures

6.1. Introduction

A choice of symplectic basis for $H_1(\Sigma, \mathbb{Z})$ gives a period matrix Ω , normalized holomorphic abelian differentials, and a spin structure \sqrt{K} ; see Appendix B. The group $\mathrm{Sp}(2g, \mathbb{Z})$ operates on $H_1(\Sigma, \mathbb{Z})$ and $H^1(\Sigma, \mathbb{Z}_2)$ and so induces an operation on spin structures. To see this action explicitly on $\mathrm{Teich}^{1/2}(\Sigma)$ we introduce the space

of quadratic refinements of cup product on $H^1(\Sigma, \mathbb{Z}_2) \rightarrow H^2(\Sigma, \mathbb{Z}_2) \approx \mathbb{Z}_2$ which are called *quarfs*. Though these objects are well known, we give an expository account of them. Following Atiyah [9], we exhibit a map from spin structures to quarfs. Both spaces are principal homogeneous spaces of $H^1(\Sigma, \mathbb{Z}_2)$ and the map commutes with the action of $H^1(\Sigma, \mathbb{Z}_2)$. Quarfs will help us compute how $\mathrm{Sp}(2g, \mathbb{Z})$ acts on $\mathrm{Teich}^{1/2}(\Sigma)$.

6.2. Quarf Primer

Let V be a \mathbb{Z}_2 vector space and let b be a nondegenerate bilinear form on V . A quadratic refinement (quarf) of b is a function $q : V \rightarrow \mathbb{Z}_2$ with the property that

$$q(v + w) - q(v) - q(w) + q(0) = b(v, w). \quad (6.1)$$

The definition implies that b is a symmetric bilinear form and $b(v, v) = 0$.

Let T_z be translation by $z \in V$ and let $q_z = q \circ T_z$. q_z is a quarf:

$$\begin{aligned} q_z(v + w) + q_z(v) + q_z(w) + q_z(0) &= q(v + w + z) + q(v + z) + q(w + z) + q(z), \\ &= \underbrace{q(v + w + z) + q(v) + q(w + z) + q(0)}_{b(v, w+z)} \\ &\quad + q(v) + \underbrace{q(w + z)}_{\text{sum cancel}} + q(0) \\ &\quad + q(v + z) + \underbrace{q(w + z)}_{\text{sum cancel}} + q(z), \\ &= b(v, w + z) + b(v, z), \\ &= b(v, w). \end{aligned}$$

Also if $\tilde{q}_r(v) = q(v) + r$ with $r \in \mathbb{Z}_2$ then \tilde{q}_r is a quarf because $4r = 0$.

Let Q_0 be the quarfs with $q(0) = 0$ and let Q_1 be the quarfs with $q(0) = 1$. Let $q_1, q_2 \in Q_0$ or let $q_1, q_2 \in Q_1$ then $\lambda = q_2 - q_1$ satisfies (1) $\lambda(0) = 0$ and (2) $\lambda(v + w) = \lambda(v) + \lambda(w)$. Over the field \mathbb{Z}_2 , the two conditions above imply that λ is a linear functional, *i.e.*, λ is in V^* , the dual space of V . Thus the number of distinct quadratic forms is $2^{\dim V}$ for both Q_0 and Q_1 . Also since b is non-degenerate there exists a $t \in V$ such that $\lambda(v) = b(v, t)$.

Theorem 6.1. *Let q be a quarf. If $q = q \circ T_w$ for some $w \in V$, then $w = 0$.*

Proof. $q(v) = (q \circ T_w)(v) = q(v + w) = q(v) + q(w) + q(0) + b(v, w)$ implying $b(v, w) = q(w) + q(0)$ for all $v \in V$. The right hand side is independent of v and the bilinear form is non-degenerate therefore $w = 0$. \square

Theorem 6.2. *Let q and q' be quarfs; then there exists a $w \in V$ such that $q'_w = q' \circ T_w = q + \epsilon$ where $\epsilon = q'(w) + q(0)$.*

Proof. $q'(v) - q'(0)$ and $q(v) - q(0)$ are in Q_0 so their difference is in V^* and given by $b(v, t)$ for some $t \in V$. Hence

$$\begin{aligned} q'_w(v) &= q'(v + w), \\ &= q'(v) + q'(w) + q'(0) + b(v, w), \\ &= (q'(0) + q(v) - q(0) + b(v, t)) + q'(w) + q'(0) + b(v, w), \\ &= q(v) + b(v, t + w) + q'(w) + q(0). \end{aligned}$$

Choose $w = t$ so that $q'_w = q + \epsilon$ where $\epsilon = q'(w) + q(0)$. \square

In general T_z does not act on Q_0 or Q_1 because $q(0)$ is not necessarily the same as $(q \circ T_z)(0) = q(z)$. This can be easily fixed by defining $\mu_z : Q_0 \rightarrow Q_0$ and $\nu_z : Q_1 \rightarrow Q_1$ by $(\mu_z q)(v) = (q \circ T_z)(v) - q(z)$ and $(\nu_z q)(v) = (q \circ T_z)(v) - q(z) + 1$. The map $\kappa : q \mapsto q - 1$ gives an isomorphism of Q_0 with Q_1 .

Theorem 6.3. *There exists a symplectic basis (α_j, β_k) for the non-degenerate bilinear form b such that $b(\alpha_j, \beta_k) = \delta_{jk}$.*

Proof. (By induction) Assume the theorem for a vector space of dimension $2(g - 1)$. Chose α_1 and find a β_1 such that $b(\alpha_1, \beta_1) = 1$. Complete α_1, β_1 to a basis for the vector space V , $\dim V = 2g$. With respect to this basis the matrix for b is of the form

$$b = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & S \\ S^t & b' \end{pmatrix}$$

Elementary row and column operations (done symmetrically and using the upper left block) can be used to define a new basis where $S = 0$. The problem is now reduced to finding a symplectic basis for b' , the $2(g - 1)$ dimensional case. \square

Choose a fixed symplectic basis (α_j, β_k) and define the basic quarf \hat{q} by

$$\hat{q}(x \cdot \alpha + y \cdot \beta) = b(x \cdot \alpha, y \cdot \beta) = x \cdot y. \quad (6.2)$$

It is easy to verify that \hat{q} is a quarf.

Theorem 6.4. *\hat{q} has $2^{g-1}(2^g + 1)$ zeroes and takes the value 1 at $2^{g-1}(2^g - 1)$ points.*

Proof. Easily proved by induction on g . Note that $(x \cdot y)_{2g} = (x \cdot y)_{2(g-1)} + x_g \cdot y_g$. If $(x \cdot y)_{2g} = 0$ then either $(x \cdot y)_{2(g-1)} = 0$ and $x_g \cdot y_g = 0$; or $(x \cdot y)_{2(g-1)} = 1$ and $x_g \cdot y_g = 1$. Therefore the total number of zeroes is $2^{g-2}(2^{g-1} + 1) \cdot 3 + 2^{g-2}(2^{g-1} - 1) \cdot 1$ as required. \square

In our case the $2g$ dimensional vector space $V = H^1(\Sigma, \mathbb{Z}_2)$. If $\xi, \eta \in H^1(\Sigma, \mathbb{Z}_2)$ let $\xi \wedge \eta$ denote the cup product. If $\mathbf{1}_2$ is the generator of $H^2(\Sigma, \mathbb{Z}_2) \approx \mathbb{Z}_2$ then we write $\xi \wedge \eta = (\xi \cup \eta)\mathbf{1}_2$. Our symplectic basis (α, β) for $H^1(\Sigma, \mathbb{Z}_2)$ is the mod 2 reduction of the dual symplectic basis in (5.4), still denoted by (α_j, β_k) . The basic quarf \hat{q} is given by

$$\hat{q}(x_j \alpha_j + y_j \beta_j) = x \cdot y = (x_j \alpha_j) \cup (y_k \beta_k) \quad (6.3)$$

The following two corollaries will connect quarfs to spin structures.

Corollary 6.5. *Let \hat{q} be the basic quarf; then every quarf q can be put into the form $q = \hat{q} \circ T_w + q(w)$ for some $w \in V$. The quarf q has $2^{g-1}(2^g + 1)$ zeroes if and only if $q(w) = 0$, and $2^{g-1}(2^g - 1)$ zeroes if and only if $q(w) = 1$.*

Proof. The first part is an immediate consequence of Theorem 6.2. The second part follows from Theorem 6.4 and the observation that $\hat{q} \circ T_w$ has the same number of zeroes as \hat{q} . \square

Corollary 6.6. *The map $w \mapsto \hat{q} \circ T_w$ is a bijection onto the set of quarfs with $2^{g-1}(2^g + 1)$ zeroes.*

Proof. Injectiveness is a consequence of Theorem 6.1 and surjectiveness is a consequence of the previous corollary. \square

This corollary can be restated: The set of quarfs with $2^{g-1}(2^g + 1)$ zeroes is a principal homogenous space for V .

6.3. Quarfs and Spin Structures

In [9], M.F. Atiyah, gives a map from spin structures (square roots of the canonical bundle K of Σ) to quadratic refinements (quarfs). Note that such square roots are a principal homogeneous space of $H^1(\Sigma, \mathbb{Z}_2)$ which is isomorphic to flat line bundles whose square is the trivial bundle. Choose a square root \mathcal{S} of K then the associated quarf $q_{\mathcal{S}}$ at w is the mod 2 index of $\bar{\partial} \otimes I_{\mathcal{S} \otimes F_w}$ where F_w is the square root of the trivial bundle determined by $w \in H^1(\Sigma, \mathbb{Z}_2)$. Atiyah shows that $q_{\mathcal{S}}$ is a quarf:

$$q_{\mathcal{S}}(w_1 + w_2) - q_{\mathcal{S}}(w_1) - q_{\mathcal{S}}(w_2) + q_{\mathcal{S}}(0) = w_1 \cup w_2.$$

For even spin structures $q_{\mathcal{S}}(0)$ takes the value 0 and for odd spin structures the value 1. Atiyah also shows that $q_{\mathcal{S}}$ has $2^{g-1}(2^g + 1)$ zeroes.

Theorem 6.7. *There is a unique even spin structure $\hat{\mathcal{S}}$ such that $q_{\hat{\mathcal{S}}} = \hat{q}$.*

Proof. Let \mathcal{S}' be any spin structure. Corollary 6.6 implies that there exists a unique w with $q_{\mathcal{S}'} = \hat{q} \circ T_w$, and that $\hat{q} = q_{\hat{\mathcal{S}}}$ where $\hat{\mathcal{S}} = \mathcal{S}' \otimes F_w$ is an even spin structure. \square

The element $w \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ determines a spin structure $\hat{\mathcal{S}} \otimes F_w$. The theta characteristic $\delta \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ associated with this spin structure is $\delta = \frac{1}{2}w$. The holomorphic section of the determinant line bundle of the $\bar{\partial}_{\hat{\mathcal{S}} \otimes F_w}$ operator is proportional to $\vartheta[\delta](0; \Omega)$.

6.4. Quarfs and Modular Transformations

In this section we describe how the symplectic group acts on spin structures. Fix a symplectic basis for $H^1(\Sigma, \mathbb{Z})$ and which gives the basic quarf on Σ . Consider the set of all spin structures over $\text{Met}(\Sigma)$ and choose a metric $\mathbf{g} \in \text{Met}(\Sigma)$. We have shown that there is a unique spin structure $\hat{\mathcal{S}}$ over \mathbf{g} with the property that the associated quarf $q_{\hat{\mathcal{S}}}$ is the basic quarf \hat{q} . The map $f \in \text{Diff}(\Sigma)$ acts on $H_1(\Sigma, \mathbb{Z})$ as a symplectic transformation in $\text{Sp}(2g, \mathbb{Z})$ and therefore induces an action on $H^1(\Sigma, \mathbb{Z}_2)$ via $\text{Sp}(2g, \mathbb{Z}_2)$. Let \mathbf{g}' be the transformed metric $f^*\mathbf{g}$. There is a unique spin structure

$\hat{\mathcal{S}}'$ over the \mathbf{g}' such that the associated quarf $q_{\hat{\mathcal{S}}'}$ is the basic quarf \hat{q} . In general⁸ $\hat{\mathcal{S}}' \neq f^*\hat{\mathcal{S}}$. To compare spin structures at different metrics we use the basic quarf to single out the reference spin structures. This argument tells us that $f^*\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}'$ differ by a square root of the trivial bundle F_t characterized by $t \in H^1(\Sigma, \mathbb{Z}_2)$:

$$f^*\hat{\mathcal{S}} = \hat{\mathcal{S}}' \otimes F_t. \quad (6.4)$$

The action on square roots of the trivial bundle is $f^*F_w = F_{f^*w}$ for $w \in H^1(\Sigma, \mathbb{Z}_2)$. Thus $f^*(\hat{\mathcal{S}} \otimes F_w) = \hat{\mathcal{S}}' \otimes (F_t \otimes F_{f^*w})$. Below we express this equation explicitly when f^* is represented by a matrix Λ .

Theorem 6.8. *If $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}_2)$, then Λ acts on spin structures and on quarfs such that $q_{\hat{\mathcal{S}}' \otimes F_{w'}}(z) = q_{\hat{\mathcal{S}} \otimes F_w}(\Lambda^{-1}z)$ where w and w' are related by*

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} (CD^t)_d \\ (AB^t)_d \end{pmatrix}, \quad (6.5)$$

and $w = \begin{pmatrix} u \\ v \end{pmatrix}$ in terms of the symplectic basis. Also $u' \cdot w' = u \cdot w$ and therefore $\text{Sp}(2g, \mathbb{Z})$ maps even (odd) spin structure to even (odd) spin structures respectively.

Proof. Under $f \in \text{Diff}(\Sigma)$, the spin structure $\hat{\mathcal{S}}' \otimes F_{w'}$ goes to a spin structure $\hat{\mathcal{S}} \otimes F_w$ and the action on $H^1(\Sigma, \mathbb{Z}_2)$ is given by the symplectic transformation $\Lambda \in \text{Sp}(2g, \mathbb{Z}_2)$. Under the action of f we have

$$q_{\hat{\mathcal{S}}' \otimes F_{w'}}(z) = q_{\hat{\mathcal{S}} \otimes F_w}(\Lambda^{-1}z) \quad (6.6)$$

as a consequence of Corollary 6.6. Letting $z = 0$ we see that the even (odd) spin structure $\hat{\mathcal{S}} \otimes F_{w'}$ maps to an even (odd) spin structure $\hat{\mathcal{S}} \otimes F_w$. The displayed equation above is equivalent to $\hat{q}(z + w') = \hat{q}(\Lambda^{-1}z + w)$ which gives relations between w, w' and Λ . The definition of quarfs gives

$$\hat{q}(z) + \hat{q}(w') + b(z, w') = \hat{q}(\Lambda^{-1}z) + \hat{q}(w) + b(\Lambda^{-1}z, w);$$

hence

$$\hat{q}(z) = \hat{q}(\Lambda^{-1}z) + b(\Lambda^{-1}z, w) - b(z, w'). \quad (6.7)$$

Replacing the argument z above by $z_1 + z_2$ and using the defining properties of quarfs gives $b(z_1, z_2) = b(\Lambda^{-1}z_1, \Lambda^{-1}z_2)$. If Λ^* is the adjoint with respect to the bilinear form b then $\Lambda^*\Lambda = I$. The equation above may be rewritten as

$$\hat{q}(z) = \hat{q}(\Lambda^{-1}z) + b(z, \Lambda w - w'). \quad (6.8)$$

If we write $z = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$\begin{aligned} \hat{q}(z) &= x \cdot y = (A^t x + C^t y) \cdot (B^t x + D^t y) + b(z, \Lambda w - w') \\ &= x \cdot y + x \cdot (AB^t)_d + y \cdot (CD^t)_d + b(z, \Lambda w - w'). \end{aligned}$$

⁸ $f^*\hat{\mathcal{S}} = \hat{\mathcal{S}}'$ if $f \in \widetilde{\text{Torelli}}(\Sigma)$; see also Section 6.5.

Since $w = \begin{pmatrix} u \\ v \end{pmatrix}$,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} (CD^t)_d \\ (AB^t)_d \end{pmatrix}.$$

Showing that $u' \cdot w' = u \cdot w$ (equivalent to the statement that odd (even) is mapped to odd (even) spin structures) is a simple algebraic computation requiring identities (6.9) and (6.10) whose proof we leave as an exercise. \square

$$0 = (AB^t)_d \cdot (CD^t)_d = (A^t C)_d \cdot (B^t D)_d. \quad (6.9)$$

$$\begin{pmatrix} (A^t C)_d \\ (B^t D)_d \end{pmatrix} = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} (DC^t)_d \\ (BA^t)_d \end{pmatrix} \quad (6.10)$$

6.5. The groups $\Gamma_2 \subset \Gamma_{1,2} \subset \text{Sp}(2g, \mathbb{Z})$

Let $\text{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ be the mod 2 reduction of $\Gamma_1 = \text{Sp}(2g, \mathbb{Z})$. We have the exact sequence

$$1 \rightarrow \Gamma_2 \rightarrow \text{Sp}(2g, \mathbb{Z}) \xrightarrow{\text{mod } 2} \text{Sp}(2g, \mathbb{Z}/2\mathbb{Z}) \rightarrow 1$$

where Γ_2 is the normal subgroup of Γ_1 given by

$$\Gamma_2 = \{\Lambda \in \Gamma_1 \mid \Lambda = I_{2g} \text{ mod } 2\}. \quad (6.11)$$

Let $\Gamma_{1,2} = \{\Lambda \in \Gamma_1 \mid \hat{q}(\Lambda z) = \hat{q}(z) \text{ mod } 2\}$ where \hat{q} is the basic quarf. Then $\Gamma_2 \subset \Gamma_{1,2} \subset \Gamma_1$, and

$$\Gamma_{1,2} = \{\Lambda \in \Gamma_1 \mid (AB^t)_d = (CD^t)_d = 0 \text{ mod } 2\}. \quad (6.12)$$

By (6.10), the conditions defining $\Gamma_{1,2}$ imply that $(A^t C)_d = (B^t D)_d = 0 \text{ mod } 2$. Note that if $f \in \text{Diff}(\Sigma)$ represents $\Lambda \in \Gamma_{1,2}$, then $f^* \hat{\mathcal{S}} = \hat{\mathcal{S}}'$, *i.e.*, F_t is the trivial bundle. See equations (6.4) and (6.5). It follows that the action of f on a quarf is a linear transformation because the translation part of (6.5) vanishes.

7. The Spinor Determinant

In our invariant the term $(\det' \bar{\partial}_\delta / \bar{N}_\delta^2)^n$ appears where δ is an odd spin structure and $\dim M = 2n$. We show that this term is related to the determinant section of the preferred spin structure and $\vartheta(0; \Omega)$; see (7.34). A discussion that applies to even spin structures may be found in [16, Section 5].

The idea is basically the following. Let $J_{g-1}(\Sigma)$ be the jacobian torus associated with line bundles with first Chern class $c_1 = g - 1$. The $\bar{\partial}$ operator coupled to those line bundles has index zero and therefore there is a determinant line bundle with a canonical section, a Quillen metric, *etc.*. The curvature of the Quillen metric is the standard translationally invariant polarization form on the torus. This means that the determinant line bundle has a unique holomorphic section. The divisor for the determinant line bundle will be a translate of the Θ divisor on the jacobian. To analyze this problem we choose a fiducial spin structure $\hat{\mathcal{S}}$, and think of $J_{g-1}(\Sigma)$

as $\{\hat{\mathcal{S}} \otimes F \mid F \in J_0(\Sigma)\}$. Equivalently a flat line bundle is equivalent to a flat connection and we can study the holomorphic family of operators $D(A) = \bar{\partial}_{\hat{\mathcal{S}}} + A^{0,1}$ where $A^{0,1} \in H^{0,1}(\Sigma)$ is a connection representing the flat line bundle F . The divisor of the section $\det D(A)$ will be the standard Θ divisor [16].

How do we define $\det D(A)$? This is discussed in detail in [16] and we provide a brief overview motivated by the path integral formulation of quantum field theory. Assume we have a chiral right moving Weyl spinor and a chiral left moving Weyl spinor coupled to our flat connection A . We know that this path integral can be regulated in a gauge invariant way yielding the answer $\det D(A)^* D(A)$. If we only had a chiral fermion coupled to the vector potential then the answer should be $\det D(A)$ (ill defined for the moment). We expect that since we can put the right and left moving systems together into the path integral that $\det(D(A)^* D(A)) = (\det D(A)^*)(\det D(A))$. The work of Quillen [6] shows that there is no holomorphic factorization and thus $\det D(A)$ cannot be a holomorphic function of $A \in H^{0,1}(\Sigma)$ if the determinant multiplication property is to be valid. In fact in [16, eq. (5.13)] it is shown that on the cover $H^{0,1}(\Sigma)$ of the jacobian $J_0(\Sigma)$, the section $\det D(z)$ is given by the theta function with general characteristic $\begin{bmatrix} u \\ v \end{bmatrix}$ where $z = \Omega u + v$:

$$\det D(z) = \left(\det \bar{\partial}(\hat{\mathcal{S}}) \right) \frac{\vartheta[z](0)}{\vartheta(0)} = \left(\det \bar{\partial}(\hat{\mathcal{S}}) \right) \frac{e^{i\pi u \cdot \Omega \cdot u + 2\pi i u \cdot v} \vartheta(\Omega u + v)}{\vartheta(0)}. \quad (7.1)$$

In the above $\det D(0) = \det \bar{\partial}(\hat{\mathcal{S}})$. Because of the exponential factor the last expression above, $\det D(z)$ is not a holomorphic function of $z = \Omega u + v$.

7.1. Bosonization Theorem for Odd Spin Structures

If an odd spin structure has semi-characteristic $\delta = \Omega a + b$, then we know that $\det D(\delta) = 0$. Consider $\det D(\delta + z)$. Note that

$$\vartheta(\delta + z) = e^{-i\pi a \cdot \Omega \cdot a - 2\pi i a \cdot (z+b)} \vartheta[\delta](z), \quad (7.2)$$

and therefore

$$\left. \frac{\partial \det D(\delta + z)}{\partial z^j} \right|_{z=0} = \frac{\det \bar{\partial}(\hat{\mathcal{S}})}{\vartheta(0)} \frac{\partial \vartheta[\delta](0)}{\partial z^j}. \quad (7.3)$$

In the above we note that since δ is an odd spin structure $4a \cdot b = 1 \pmod{2}$.

To connect with $\det' \bar{\partial}_{\delta} / \bar{N}_{\delta}^2$ consider $(\det D^* D)(\delta + z)$ in perturbation theory near $z = 0$. Let $\lambda(z)$ be the smallest eigenvalue. Since $\lambda(0) = 0$,

$$\lim_{z \rightarrow 0} \frac{(\det D^* D)(\delta + z)}{\lambda(z)} = \det' \bar{\partial}_{\delta}^* \bar{\partial}_{\delta} \text{ by definition.} \quad (7.4)$$

The best way to compute the left hand side above is to change viewpoint temporarily and think of $D(z + \delta)$ as a translate of $\bar{\partial}_{\delta}$ by a flat connection. Thus we define $\mathcal{D}(z) = \bar{\partial}_{\delta} + A^{0,1}$ acting on sections of the line bundle \mathcal{S}_{δ} associated with the odd semi-characteristic δ . Let $\phi(z)$ be the eigensection of $\mathcal{D}^* \mathcal{D}(z)$ with eigenvalue $\lambda(z)$. Note that $\phi(0) = h_{\delta}$ which is a holomorphic section of $\bar{\partial}_{\delta}$. We now do perturbation theory. Begin with the equation

$$\mathcal{D}^*(z) \mathcal{D}(z) \phi(z) = \lambda(z) \phi(z) \quad (7.5)$$

and differentiate:

$$\mathcal{D}^* \frac{\partial \mathcal{D}}{\partial \bar{z}^i} \phi + \mathcal{D}^* \mathcal{D} \frac{\partial \phi}{\partial z^i} = \frac{\partial \lambda}{\partial z^i} \phi + \lambda \frac{\partial \phi}{\partial z^i}, \quad (7.6)$$

$$\frac{\partial \mathcal{D}^*}{\partial \bar{z}^j} \mathcal{D} \phi + \mathcal{D}^* \mathcal{D} \frac{\partial \phi}{\partial \bar{z}^j} = \frac{\partial \lambda}{\partial \bar{z}^j} \phi + \lambda \frac{\partial \phi}{\partial \bar{z}^j}. \quad (7.7)$$

Evaluating the above at $z = 0$ gives

$$\bar{\partial}_\delta^* \frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta + \bar{\partial}_\delta^* \bar{\partial}_\delta \frac{\partial \phi(0)}{\partial z^i} = \frac{\partial \lambda(0)}{\partial z^i} h_\delta, \quad (7.8)$$

$$\bar{\partial}_\delta^* \bar{\partial}_\delta \frac{\partial \phi(0)}{\partial \bar{z}^j} = \frac{\partial \lambda(0)}{\partial \bar{z}^j} h_\delta. \quad (7.9)$$

Taking the inner product with h_δ gives

$$0 = \frac{\partial \lambda(0)}{\partial z^i} = \frac{\partial \lambda(0)}{\partial \bar{z}^j}. \quad (7.10)$$

Insert this into the previous equations to obtain

$$\left. \frac{\partial}{\partial z^i} (\mathcal{D} \phi) \right|_{z=0} = \frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta + \bar{\partial}_\delta \frac{\partial \phi(0)}{\partial z^i} \in \ker \bar{\partial}_\delta^*, \quad (7.11)$$

$$\frac{\partial \phi(0)}{\partial \bar{z}^j} \in \ker \bar{\partial}_\delta. \quad (7.12)$$

To show that $\partial^2 \lambda(0) / \partial z^i \partial z^j = 0$, differentiate (7.6) one more time and obtain

$$\mathcal{D}^* \frac{\partial \mathcal{D}}{\partial z^i} \frac{\partial \phi}{\partial z^j} + \mathcal{D}^* \frac{\partial \mathcal{D}}{\partial z^j} \frac{\partial \phi}{\partial z^i} + \mathcal{D}^* \mathcal{D} \frac{\partial^2 \phi}{\partial z^j \partial z^i} = \frac{\partial^2 \lambda}{\partial z^j \partial z^i} \phi + \frac{\partial \lambda}{\partial z^i} \frac{\partial \phi}{\partial z^j} + \frac{\partial \lambda}{\partial z^j} \frac{\partial \phi}{\partial z^i} + \lambda \frac{\partial^2 \phi}{\partial z^j \partial z^i}.$$

Evaluate this expression at $z = 0$ and take the inner product with h_δ to get the desired result. Note that since the eigenvalue $\lambda(z)$ is real, $\partial^2 \lambda(0) / \partial \bar{z}^i \partial \bar{z}^j = 0$, easily corroborated by differentiating (7.7).

Finally we consider the term $\partial^2 \lambda(0) / \partial z^i \partial \bar{z}^j$. Differentiate (7.6) with respect to \bar{z}^j and obtain

$$\begin{aligned} & \frac{\partial \mathcal{D}^*}{\partial \bar{z}^j} \frac{\partial \mathcal{D}}{\partial z^i} \phi + \mathcal{D}^* \frac{\partial \mathcal{D}}{\partial z^i} \frac{\partial \phi}{\partial \bar{z}^j} + \frac{\partial \mathcal{D}^*}{\partial \bar{z}^j} \mathcal{D} \frac{\partial \phi}{\partial z^i} + \mathcal{D}^* \mathcal{D} \frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i} \\ &= \frac{\partial^2 \lambda}{\partial \bar{z}^j \partial z^i} \phi + \frac{\partial \lambda}{\partial z^i} \frac{\partial \phi}{\partial \bar{z}^j} + \frac{\partial \lambda}{\partial \bar{z}^j} \frac{\partial \phi}{\partial z^i} + \lambda \frac{\partial^2 \phi}{\partial \bar{z}^j \partial z^i}. \end{aligned} \quad (7.13)$$

Evaluating at $z = 0$ and regrouping terms gives

$$\frac{\partial \mathcal{D}^*(0)}{\partial \bar{z}^j} \left(\frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta + \bar{\partial}_\delta \frac{\partial \phi(0)}{\partial z^i} \right) + \bar{\partial}_\delta^* \frac{\partial \mathcal{D}(0)}{\partial z^i} \frac{\partial \phi(0)}{\partial \bar{z}^j} + \bar{\partial}_\delta^* \bar{\partial}_\delta \frac{\partial^2 \phi(0)}{\partial \bar{z}^j \partial z^i} = \frac{\partial^2 \lambda(0)}{\partial \bar{z}^j \partial z^i} h_\delta. \quad (7.14)$$

Taking the inner product with h_δ gives

$$\left\langle h_\delta, \frac{\partial \mathcal{D}^*(0)}{\partial \bar{z}^j} \left(\frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta + \bar{\partial}_\delta \frac{\partial \phi(0)}{\partial z^i} \right) \right\rangle = \frac{\partial^2 \lambda(0)}{\partial \bar{z}^j \partial z^i} \langle h_\delta, h_\delta \rangle. \quad (7.15)$$

Let $\psi_\delta \in \ker \bar{\partial}_\delta^*$ then (7.11) tells us that there exist constants $r_i \in \mathbb{C}$ such that

$$\frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta + \bar{\partial}_\delta \frac{\partial \phi(0)}{\partial z^i} = r_i \psi_\delta. \quad (7.16)$$

Taking the inner product with ψ_δ and using $\bar{\partial}_\delta^* \psi_\delta = 0$,

$$\left\langle \psi_\delta, \frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta \right\rangle = r_i \langle \psi_\delta, \psi_\delta \rangle. \quad (7.17)$$

Next we manipulate (7.15):

$$\begin{aligned} \frac{\partial^2 \lambda(0)}{\partial \bar{z}^j \partial z^i} \langle h_\delta, h_\delta \rangle &= \left\langle h_\delta, \frac{\partial \mathcal{D}^*(0)}{\partial \bar{z}^j} \left(\frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta + \bar{\partial}_\delta \frac{\partial \phi(0)}{\partial z^i} \right) \right\rangle, \\ &= \left\langle \frac{\partial \mathcal{D}(0)}{\partial z^j} h_\delta, \left(\frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta + \bar{\partial}_\delta \frac{\partial \phi(0)}{\partial z^i} \right) \right\rangle, \\ &= \left\langle r_j \psi_\delta - \bar{\partial}_\delta \frac{\partial \phi(0)}{\partial z^j}, r_i \psi_\delta \right\rangle, \\ &= \bar{r}_j r_i \langle \psi_\delta, \psi_\delta \rangle, \end{aligned}$$

where we used $\bar{\partial}_\delta^* \psi_\delta = 0$. We rewrite this as

$$\frac{\partial^2 \lambda(0)}{\partial \bar{z}^j \partial z^i} = \frac{1}{\langle h_\delta, h_\delta \rangle \langle \psi_\delta, \psi_\delta \rangle} \overline{\left\langle \psi_\delta, \frac{\partial \mathcal{D}(0)}{\partial z^j} h_\delta \right\rangle} \left\langle \psi_\delta, \frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta \right\rangle. \quad (7.18)$$

Note that as required, the answer is independent of how we scale h_δ and ψ_δ and that the second derivative factorizes. We now derive a simple formula for (7.18).

Observe that $\bar{\partial}_\delta : \Lambda^{1/2,0}(\Sigma) \rightarrow \Lambda^{1/2,1}(\Sigma) \approx (\Lambda^{1/2,0}(\Sigma))^*$ and that the hermitian inner product on our line bundles gives us a conjugate linear isomorphism between $\Lambda^{1/2,0}(\Sigma)$ and $\Lambda^{1/2,1}(\Sigma)$. Hence we can identify h_δ with ψ_δ and

$$\begin{aligned} \left\langle \psi_\delta, \frac{\partial \mathcal{D}(0)}{\partial z^i} h_\delta \right\rangle &= \frac{1}{2i} \int_\Sigma \frac{\partial \mathcal{D}(0)}{\partial z^i} \wedge h_\delta^2, \\ &= \frac{1}{2i} \int_\Sigma 2\pi i (\Omega - \bar{\Omega})_{ij}^{-1} \bar{\omega}_j \wedge \sum_k \frac{\partial \vartheta[\delta](0)}{\partial z^k} \omega_k \\ &= \pi \frac{\partial \vartheta[\delta](0)}{\partial z^i} \end{aligned}$$

Thus (7.18) becomes

$$\frac{\partial^2 \lambda(0)}{\partial \bar{z}^j \partial z^i} = \pi^2 \frac{1}{\langle h_\delta, h_\delta \rangle^2} \frac{\overline{\partial \vartheta[\delta](0)}}{\partial z^j} \frac{\partial \vartheta[\delta](0)}{\partial z^i}. \quad (7.19)$$

[16, eq. (5.11)] states

$$\det D(u)^* D(u) = \frac{\det \bar{\partial}_\delta^* \bar{\partial}_\delta}{|\vartheta(0)|^2} |\vartheta(u)|^2 e^{i\pi(u-\bar{u})(\Omega-\bar{\Omega})^{-1}(u-\bar{u})}. \quad (7.20)$$

Letting $u = \delta + z$ we find

$$\det D(u)^* D(u) = \frac{\det \bar{\partial}_{\mathcal{S}}^* \bar{\partial}_{\mathcal{S}}}{|\vartheta(0)|^2} \left| e^{-i\pi a \cdot \Omega \cdot a - 2\pi i a \cdot (z+b)} \vartheta[\delta](z) \right|^2 e^{i\pi(u-\bar{u}) \cdot (\Omega - \bar{\Omega})^{-1} \cdot (u-\bar{u})}. \quad (7.21)$$

The Taylor series of the above is

$$\begin{aligned} \det D(u)^* D(u) &= \frac{\det \bar{\partial}_{\mathcal{S}}^* \bar{\partial}_{\mathcal{S}}}{|\vartheta(0)|^2} \left| e^{-i\pi a \cdot \Omega \cdot a} \sum_j z^j \frac{\partial \vartheta[\delta](0)}{\partial z^j} \right|^2 e^{i\pi a \cdot (\Omega - \bar{\Omega}) \cdot a} + O(z^3), \\ &= \frac{\det \bar{\partial}_{\mathcal{S}}^* \bar{\partial}_{\mathcal{S}}}{|\vartheta(0)|^2} \left| \sum_j z^j \frac{\partial \vartheta[\delta](0)}{\partial z^j} \right|^2 + O(z^3). \end{aligned} \quad (7.22)$$

Since for a normalized spinor \hat{h}_δ we defined $h_\delta = N_\delta \hat{h}_\delta$ and therefore $|N_\delta|^2 = \langle h_\delta, h_\delta \rangle$ we have

$$\pi^2 \frac{\det' \bar{\partial}_\delta^* \bar{\partial}_\delta}{|\langle h_\delta, h_\delta \rangle|^2} = \pi^2 \frac{\det' \bar{\partial}_\delta^* \bar{\partial}_\delta}{|N_\delta|^4} = \frac{\det \bar{\partial}_{\mathcal{S}}^* \bar{\partial}_{\mathcal{S}}}{|\vartheta(0)|^2}. \quad (7.23)$$

This formula says that if δ and ϵ are odd spin structures then

$$\frac{\det' \bar{\partial}_\delta^* \bar{\partial}_\delta}{|\langle h_\delta, h_\delta \rangle|^2} = \frac{\det' \bar{\partial}_\epsilon^* \bar{\partial}_\epsilon}{|\langle h_\epsilon, h_\epsilon \rangle|^2}. \quad (7.24)$$

In other words the function $\det'(\bar{\partial}_\delta^* \bar{\partial}_\delta)/|\langle h_\delta, h_\delta \rangle|^2$ on $\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$ descends to a function on $\text{Teich}(\Sigma)$.

This leads to a new result about bosonization for odd spin structures. The standard bosonization result [17] is that for every *even* spin structure β

$$\left(\frac{\text{vol } \Sigma \det H}{\det_{1^\perp} \Delta_0} \right)^{1/2} = \frac{1}{2} \frac{\det \bar{\partial}_\beta^* \bar{\partial}_\beta}{|\vartheta[\beta](0)|^2}. \quad (7.25)$$

If we compare this with (7.23) we see that for every *odd* spin structure δ

$$\left(\frac{\text{vol } \Sigma \det H}{\det_{1^\perp} \Delta_0} \right)^{1/2} = \frac{\pi^2}{2} \frac{\det' \bar{\partial}_\delta^* \bar{\partial}_\delta}{|\langle h_\delta, h_\delta \rangle|^2}. \quad (7.26)$$

7.2. Trivializing

We discuss the trivialization of the determinant line bundle for the Dirac operator over spin Teichmuller space $\text{Teich}^{1/2}(\Sigma)$ and in the process we clarify what we mean by $\det \bar{\partial}$ throughout much of this article. $\text{Teich}(\Sigma)$ is a convex space with the topology of $(\mathbb{R} \times \mathbb{R}_+)^{3g-3}$ and therefore line bundles are trivializable. Spin Teichmuller space $\text{Teich}^{1/2}(\Sigma)$ and spin moduli space $\mathcal{M}^{1/2}(\Sigma)$ are respectively finite covers of Teichmuller space $\text{Teich}(\Sigma)$ and moduli space $\mathcal{M}(\Sigma)$ with projection maps π and $\pi_{1/2}$. They are related by the following commutative diagram:

$$\begin{array}{ccc} \text{Teich}^{1/2}(\Sigma) & \xrightarrow{\pi_{1/2}} & \mathcal{M}^{1/2}(\Sigma) \\ \text{forgetful} \downarrow & & \downarrow \text{forgetful} \\ \text{Teich}(\Sigma) & \xrightarrow{\pi} & \mathcal{M}(\Sigma) \end{array} \quad (7.27)$$

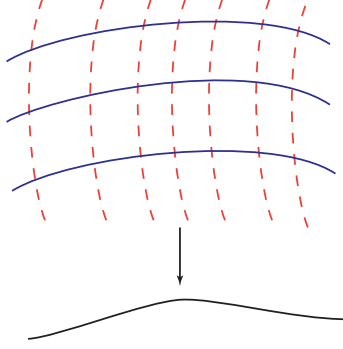


Figure 1: The above is a schematic figure of spin Teichmuller space, $\text{Teich}^{1/2}(\Sigma)$, and how it fits inside of the bundle $\text{Teich}(\Sigma) \times J_0(\Sigma)$. The base is $\text{Teich}(\Sigma)$ and the fibers are isomorphic to $J_0(\Sigma)$. The preferred spin structure $\hat{\mathcal{S}}$ associated with the basic quarf is taken to be the origin of the jacobian and all the other spin structures correspond to the half period points of the torus. Spin Teichmuller space is represented by the horizontal leaves.

The vertical arrows correspond to forgetful functors that ignore the spin structure information.

Consider the conformal field theory of two chiral $(1/2, 0)$ fermions⁹. According to Friedan and Shenker [10], the expectation value of the energy momentum tensor $\langle T_{zz} \rangle (dz)^2$ of that conformal field theory gives a flat hermitian connection \mathcal{A} on a holomorphic line bundle over spin moduli space $\mathcal{M}^{1/2}(\Sigma)$. The line bundle in question $\mathcal{L} \rightarrow \mathcal{M}^{1/2}(\Sigma)$ is the determinant line bundle associated with the Dirac operator and the “partition function” is the determinant section. We can pull back the line bundle and connection to the cover $\text{Teich}^{1/2}(\Sigma)$.

Restrict to the leaf $\text{Teich}^{1/2}(\Sigma, \hat{q})$ in $\text{Teich}^{1/2}(\Sigma)$ associated with the basic quarf \hat{q} . On this leaf we have the determinant line bundle $\tilde{\mathcal{L}}(\hat{q}) \rightarrow \text{Teich}^{1/2}(\Sigma, \hat{q})$ of the preferred spin structure $\hat{\mathcal{S}}$. Next we trivialize this line bundle. The pulled back expectation value of the energy momentum tensor gives a flat holomorphic connection $\tilde{\mathcal{A}}$ on the determinant line bundle $\tilde{\mathcal{L}}(\hat{q}) \rightarrow \text{Teich}^{1/2}(\Sigma, \hat{q})$. Pick a distinguished point $\hat{\mathcal{S}}_0 \in \text{Teich}^{1/2}(\Sigma, \hat{q})$ and a reference point $\hat{\sigma}_0$ on the fiber of $\tilde{\mathcal{L}}_{\hat{\mathcal{S}}_0}$ over $\hat{\mathcal{S}}_0$. Once we have picked $\hat{\sigma}_0$ we can identify the fiber $\tilde{\mathcal{L}}_{\hat{\mathcal{S}}_0}$ with \mathbb{C} . The trivialization is along the discussion in Section 4.1. Let σ be a point on the fiber over $\mathcal{S} \in \text{Teich}^{1/2}(\Sigma, \hat{q})$ and parallel transport σ along any curve from \mathcal{S} to $\hat{\mathcal{S}}_0$. The choice of curve is irrelevant because $\text{Teich}^{1/2}(\Sigma, \hat{q})$ is contractible and the connection is flat. The trivialization map $\phi : \tilde{\mathcal{L}}(\hat{q}) \rightarrow \text{Teich}^{1/2}(\Sigma, \hat{q}) \times \mathbb{C}$ is given by

$$\sigma(\mathcal{S}) = (\phi\sigma)(\mathcal{S}) \exp \left(- \int_{\hat{\mathcal{S}}_0}^{\mathcal{S}} \tilde{\mathcal{A}} \right) \hat{\sigma}_0 . \quad (7.28)$$

⁹With an even number of chiral fermions we can frame the discussion in terms of determinant line bundles and not pfaffian line bundles.

It follows from the definition of spin Teichmuller space that $\text{Teich}^{1/2}(\Sigma, \hat{q})$ is isomorphic to $\text{Teich}(\Sigma)$ and we implicitly think of the trivialized line bundle as being $\text{Teich}(\Sigma) \times \mathbb{C}$.

We trivialize $\tilde{\mathcal{L}} \rightarrow \text{Teich}^{1/2}(\Sigma)$ by embedding $\text{Teich}^{1/2}(\Sigma)$ in $\text{Teich}(\Sigma) \times J_0(\Sigma)$ as depicted in Figure 1. Fix a point $\tau \in \text{Teich}(\Sigma)$ and consider the family of operators $\bar{\partial}(\hat{\mathcal{S}} \otimes F)$ where $F \in J_0(\Sigma)$ is a flat line bundle and $\hat{\mathcal{S}}$ is the spin structure associated with the basic quarf \hat{q} . By going to the cover $H^{0,1}(\Sigma)$ of $J_0(\Sigma)$ we can trivialize the family using a parallel transport method analogous to the one discussed in Section 4.1. We can restrict to the half lattice points in $H^{0,1}(\Sigma)$ and we obtain that if $\delta = \frac{1}{2}w$ where $w \in H(\Sigma, \mathbb{Z}/2\mathbb{Z})$ then

$$\det \bar{\partial}(\hat{\mathcal{S}} \otimes F_w) = \frac{\vartheta[\delta](0; \Omega)}{\vartheta(0; \Omega)} \det \bar{\partial}(\hat{\mathcal{S}}), \quad (7.29)$$

according to (7.1). For the moment $\det \bar{\partial}(\hat{\mathcal{S}})$ is a point on the fiber of the determinant line at $\hat{\mathcal{S}} \in \text{Teich}^{1/2}(\Sigma, \hat{q})$ over the point $\tau \in \text{Teich}(\Sigma)$. Next we use the trivialization of $\tilde{\mathcal{L}}(\hat{q}) \rightarrow \text{Teich}^{1/2}(\Sigma, \hat{q})$ previously described and this is how we interpret the determinant section. Since we have the Quillen metric, $\det \bar{\partial}(\hat{\mathcal{S}})$ is known up to a phase.

Our conformal field theory of chiral spinors is associated with the determinant line bundle $\mathcal{L} \rightarrow \mathcal{M}^{1/2}(\Sigma)$ with holomorphic partition section Z . This holomorphic section satisfies the parallel transport equation $\partial_{\mathcal{M}^{1/2}(\Sigma)} Z + \mathcal{A}Z = 0$, and can be pulled back to a section \tilde{Z} of $\tilde{\mathcal{L}} \rightarrow \text{Teich}^{1/2}(\Sigma)$ where it satisfies $\partial_{\text{Teich}^{1/2}(\Sigma)} \tilde{Z} + \tilde{\mathcal{A}}\tilde{Z} = 0$. In the trivialization just described, \tilde{Z} becomes a function that we denote by $\det \bar{\partial}$. Note that after restriction to $\text{Teich}^{1/2}(\Sigma, \hat{q})$, the holomorphic determinant section $\det \bar{\partial}(\hat{\mathcal{S}})$ satisfies the parallel transport equation $\partial_{\text{Teich}(\Sigma)} \det \bar{\partial} + \tilde{\mathcal{A}} \det \bar{\partial} = 0$.

Let $\mathcal{D} \rightarrow \text{Teich}^{1/2}(\Sigma)$ be the trivialized determinant line bundle as described in the previous paragraphs. If $f \in \text{Diff}(\Sigma)$ represent $\Lambda \in \text{Sp}(2g, \mathbb{Z})$ then $f^*\mathcal{D} = \mathcal{D} \otimes \mathcal{C}$ where $\mathcal{C} \rightarrow \text{Teich}^{1/2}(\Sigma)$ is a circle bundle because f acts isometrically. Consider the section $\det \bar{\partial}$ of \mathcal{D} . Since f acts by isometries, $f^*(\det \bar{\partial})_{f^*\mathcal{D}} = u \cdot (\det \bar{\partial})_{\mathcal{D}}$ where u is a section of \mathcal{C} and $|u| = 1$. Since the determinant sections vary holomorphically over $\text{Teich}^{1/2}(\Sigma)$, u is a constant section. The dependence of u on f is only through Λ since we have already taken into account the action of the normal subgroup $\text{Diff}_0(\Sigma)$.

7.3. Defining $\det' \bar{\partial}$

Having trivialized the bundle, we can define $\det' \bar{\partial}$ by exploiting the embedding of $\text{Teich}^{1/2}(\Sigma)$ in $\text{Teich}(\Sigma) \times J_0(\Sigma)$. We rewrite (7.1) as

$$\begin{aligned} \det D(\delta + z) &= \frac{\det \bar{\partial}(\hat{\mathcal{S}})}{\vartheta(0)} e^{i\pi(a+u) \cdot \Omega \cdot (a+u) + 2\pi i(a+u) \cdot (b+v)} \vartheta(\delta + z), \\ &= \frac{\det \bar{\partial}(\hat{\mathcal{S}})}{\vartheta(0)} e^{i\pi(a+u) \cdot \Omega \cdot (a+u) + 2\pi i(a+u) \cdot (b+v)} \\ &\quad \times e^{-i\pi a \cdot \Omega a - 2\pi i u \cdot (z+b)} \vartheta[\delta](z) \\ &= \frac{\det \bar{\partial}(\hat{\mathcal{S}})}{\vartheta(0)} e^{i\pi u \cdot \Omega u + 2\pi i u \cdot (v+b)} \vartheta[\delta](z). \end{aligned} \quad (7.30)$$

From (7.19) we have

$$\lambda(z) = \left| \frac{\pi}{N_\delta^2} \sum_{j=i}^g z^j \frac{\partial \vartheta[\delta](0)}{\partial z^j} \right|^2 + O(z^3), \quad (7.31)$$

and we can take an approximate “holomorphic square root”

$$\sqrt{\lambda(z)} = e^{i\psi} \frac{\pi}{N_\delta^2} \sum_{j=i}^g z^j \frac{\partial \vartheta[\delta](0)}{\partial z^j} + O(z^2). \quad (7.32)$$

In the above we introduced an arbitrary phase ψ . The subleading terms will not have holomorphicity properties. Thus we can define $\det' \bar{\partial}_\delta$ by mimicking¹⁰ (7.4) and defining

$$\det' \bar{\partial}_\delta = \lim_{z \rightarrow 0} \frac{D(\delta + z)}{\sqrt{\lambda(z)}}. \quad (7.33)$$

We compute the above with the result

$$\frac{\det' \bar{\partial}_\delta}{N_\delta^2} = e^{-i\psi} \frac{1}{\pi} \frac{\det \bar{\partial}(\hat{\mathcal{S}})}{\vartheta(0)}.$$

The arbitrary phase in the normalization factor N_δ can be chosen so that $e^{i\psi} = 1$ and

$$\frac{\det' \bar{\partial}_\delta}{N_\delta^2} = \frac{1}{\pi} \frac{\det \bar{\partial}(\hat{\mathcal{S}})}{\vartheta(0)}. \quad (7.34)$$

This agrees with explicit results in genus one.

8. Geometric Symplectic Action on Determinants

8.1. Quillen Isometry

The determinant for the Dirac laplacian is given by

$$(\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}} \otimes F_w) = (\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}}) \left| \frac{\vartheta[\delta](0; \Omega)}{\vartheta(0; \Omega)} \right|^2. \quad (8.1)$$

where $\hat{\mathcal{S}}$ is the spin structure described by the basic quarf \hat{q} and $\delta = \frac{1}{2}w \bmod \mathbb{Z}$, see (7.21).

Let $\Lambda \in \mathrm{Sp}(2g, \mathbb{Z})$ and let $f \in \mathrm{Diff}(\Sigma)$ be a representative for Λ . Because of the functorial properties of $\bar{\partial}$, the eigenvalues are preserved under the action of f and we conclude that

$$(\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}}' \otimes F_t \otimes F_{f^*w}) = (\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}} \otimes F_w). \quad (8.2)$$

¹⁰This is what the path integral suggests as a possible definition.

Remark 8.1. Equation (8.2) states that the diffeomorphism f acts as an isometry with respect to the Quillen metric on the f -related determinant line bundles.

Remark 8.2. The Quillen isometry is more general: if F_w is replaced by an arbitrary flat line bundle F_χ corresponding to character χ , then (8.2) is still valid.

Using the notation that $F_{w'} = F_{f^*w} \otimes F_t$, $\delta' = \frac{1}{2}w'$, etc., a straightforward computation gives

$$\begin{aligned} (\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}}' \otimes F_{w'}) &= (\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}}') \left| \frac{\vartheta[\delta'](0; \Omega')}{\vartheta(0; \Omega')} \right|^2, \\ &= (\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}}') \left| \frac{\varepsilon(\Lambda) e^{-i\pi\phi(\delta)} \det(C\Omega + D)^{1/2} \vartheta[\delta](0; \Omega)}{\varepsilon(\Lambda) \det(C\Omega + D)^{1/2} \vartheta(0; \Omega)} \right|^2, \\ &= |\det(C\Omega + D)| |\vartheta[\delta](0; \Omega)|^2 \frac{(\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}}')}{|\vartheta(0; \Omega')|^2}. \end{aligned}$$

Hence

$$\frac{(\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}}')}{|\vartheta(0; \Omega')|^2} = |\det(C\Omega + D)| \frac{(\det \bar{\partial}^* \bar{\partial})(\hat{\mathcal{S}}')}{|\vartheta(0; \Omega)|^2}. \quad (8.3)$$

8.2. Determinant Section for Spin Bundles

In this section we study in more detail the Quillen isometry. Remark 8.2 tells us that in general we expect a relationship

$$(\det \bar{\partial})(\hat{\mathcal{S}}' \otimes F_t \otimes f^* F_\chi) = e^{i\xi(\chi)} (\det \bar{\partial})(\hat{\mathcal{S}} \otimes F_\chi).$$

A difficulty with the above is $\hat{\mathcal{S}} \otimes F_\chi$ is an odd spin structure then the determinant sections vanish and it is not clear that the phase can be defined. In this section we develop a limit process that uniquely determines the phase for an odd spin structure.

Fix a symplectic basis for $H_1(\Sigma, \mathbb{Z})$. Let F_χ be the flat line bundle specified by character χ . We use F_χ as a perturbation and assume it is close to the trivial bundle. Identify the character χ with a point ζ near the origin of \mathbb{C}^g . The assignment of ζ is given by the standard coordinates on $H^{0,1}(\Sigma)$, the cover of $J_0(\Sigma)$. We abuse notation and denote F_χ by F_ζ . Let $\hat{\mathcal{S}} \otimes F_w$ be an odd spin structure, $w \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$. Trivializing the determinant line bundle over $H^{0,1}(\Sigma)$ for the family of $\bar{\partial}$ operators acting on $\hat{\mathcal{S}} \otimes F_\zeta$ gives

$$(\det \bar{\partial})(\hat{\mathcal{S}} \otimes F_w \otimes F_\zeta) = (\det \bar{\partial})(\hat{\mathcal{S}}) \frac{\vartheta[\delta + \zeta](0; \Omega)}{\vartheta(0; \Omega)}. \quad (8.4)$$

using¹¹ (7.3) and (7.2).

¹¹The algebraic geometry ϑ -function conventions use semi-characteristics $\delta = \frac{1}{2}w \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. At times we abuse notation using w and δ interchangeably. Shifting δ, ζ by a lattice vector may lead to phases in the formula above.

Let $f \in \text{Diff}(\Sigma)$ that represents $\Lambda \in \text{Sp}(2g, \mathbb{Z})$. Since f acts as an isometry on the determinant line bundles, see Section 8.1, we obtain

$$(\det \bar{\partial})(\hat{\mathcal{S}}' \otimes F_t \otimes F_{f^*w} \otimes F_{f^*z}) = e^{i\xi(w+z;\Lambda)} (\det \bar{\partial})(\hat{\mathcal{S}} \otimes F_w \otimes F_z), \quad (8.5)$$

where ξ is a phase and F_t is the “translational shift” flat line bundle discussed previously.

Observe that

$$\begin{aligned} e^{i\xi(0,\Lambda)} \frac{(\det \bar{\partial})(\hat{\mathcal{S}})}{\vartheta(0;\Omega)} &= \frac{(\det \bar{\partial})(\hat{\mathcal{S}}' \otimes F_t)}{\vartheta(0;\Omega)} \\ &= \frac{\vartheta[t](0;\Omega')}{\vartheta(0;\Omega)} \frac{(\det \bar{\partial})(\hat{\mathcal{S}}')}{\vartheta(0;\Omega')} \\ &= \varepsilon(\Lambda) e^{-i\pi\phi(0,\Lambda)} \det(C\Omega + D)^{1/2} \frac{(\det \bar{\partial})(\hat{\mathcal{S}}')}{\vartheta(0;\Omega')}, \\ &= \varepsilon(\Lambda) \det(C\Omega + D)^{1/2} \frac{(\det \bar{\partial})(\hat{\mathcal{S}}')}{\vartheta(0;\Omega')}, \end{aligned} \quad (8.6)$$

where $\phi(0, \Lambda) = 0$ by (5.10).

Remark 8.3. In genus one, fermionic determinants are of the form $\vartheta[\delta](\tau)/\eta(\tau)$, therefore $\det/\vartheta = 1/\eta$ is universal for all even spin structures. This is a special case of $(\det \bar{\partial})(\hat{\mathcal{S}} \otimes F_w)/\vartheta[\delta](0) = (\det \bar{\partial})(\hat{\mathcal{S}})/\vartheta(0)$. Under the modular transformation $T(\tau) = \tau' = \tau + 1$ we have that $(\det \bar{\partial})(\hat{\mathcal{S}}' \otimes F_t) = \vartheta_{01}(0; \tau')/\eta(\tau')$ and $(\det \bar{\partial})(\hat{\mathcal{S}}) = \vartheta_{00}(0; \tau)/\eta(\tau)$. Note that $\vartheta_{01}(0; \tau')/\eta(\tau') = e^{-2\pi i/24} \vartheta_{00}(0; \tau)/\eta(\tau)$ and so we conclude that $e^{i\xi(0,T)} = e^{-2\pi i/24}$ in this example.

Remark 8.4. According to (8.6), $\vartheta(0;\Omega)/\det \bar{\partial}(\hat{\mathcal{S}})$ is a modular form of weight $1/2$. It is constant along the fibers in the covering $\text{Teich}^{1/2}(\Sigma) \rightarrow \text{Teich}(\Sigma)$ in the sense that for even spin structures we use result (7.29) and for odd spin structures we use (7.34). For this reason we can think of $\vartheta(0;\Omega)/\det \bar{\partial}(\hat{\mathcal{S}})$ as defined on $\text{Teich}(\Sigma)$ as a higher genus generalization of the Dedekind η -function.

Remark 8.5. Even in genus one, the computation of the modular transformations properties of the Dedekind η -function using the geometry of determinant line bundles is a very technical exercise [18].

If in (8.5), $\hat{\mathcal{S}} \otimes F_w$ is an odd spin structure and $z = 0$ then we expect that $e^{i\xi(w;\Lambda)}$ is not well defined because the determinant sections vanish. Below, the flat bundle F_z in (8.5) is used to determine $e^{i\xi(w;\Lambda)}$ via a limit process for $\hat{\mathcal{S}} \otimes F_w$ an odd spin structure. First we note that

$$\begin{aligned} \frac{(\det' \bar{\partial})(\hat{\mathcal{S}} \otimes F_\delta)}{N_\delta^2} &= \lim_{z \rightarrow 0} \frac{(\det \bar{\partial})(\hat{\mathcal{S}} \otimes F_\delta \otimes F_z)}{N_\delta^2 \sqrt{\lambda(z)}}, \\ &= \frac{1}{\pi} \frac{(\det \bar{\partial})(\hat{\mathcal{S}})}{\vartheta(0;\Omega)}, \end{aligned} \quad (8.7)$$

where we used (7.34). Next, using the notation $F_{w'} = F_t \otimes F_{f^*w}$ and $F_{z'} = F_{f^*z}$ we compute

$$\begin{aligned}
\frac{(\det' \bar{\partial})(\hat{\mathcal{S}}' \otimes F_{w'})}{N_{\delta'}^2} &= \lim_{z' \rightarrow 0} \frac{(\det \bar{\partial})(\hat{\mathcal{S}}' \otimes F_{w'} \otimes F_{z'})}{N_{\delta'}^2 \sqrt{\lambda'(z')}} , \\
&= \lim_{z' \rightarrow 0} \frac{e^{i\xi(\delta, \Lambda)} (\det \bar{\partial})(\hat{\mathcal{S}} \otimes F_w \otimes F_z)}{\pi \sum z'_j \partial \vartheta[\delta'](0; \Omega') / \partial z'_j} , \\
&= \lim_{z \rightarrow 0} \frac{e^{i\xi(\delta, \Lambda)} (\det \bar{\partial})(\hat{\mathcal{S}} \otimes F_w \otimes F_z)}{\pi \varepsilon(\Lambda) e^{-i\pi\phi(\delta, \Lambda)} \det(C\Omega + D)^{1/2} \sum z_j \partial \vartheta[\delta](0; \Omega) / \partial z_j} , \\
&= \frac{e^{i\xi(\delta, \Lambda)}}{\varepsilon(\Lambda) e^{-i\pi\phi(\delta, \Lambda)} \det(C\Omega + D)^{1/2}} \lim_{z \rightarrow 0} \frac{(\det \bar{\partial})(\hat{\mathcal{S}} \otimes F_w \otimes F_z)}{N_\delta^2 \sqrt{\lambda(z)}} , \\
&= \frac{e^{i\xi(\delta, \Lambda)}}{\varepsilon(\Lambda) e^{-i\pi\phi(\delta, \Lambda)} \det(C\Omega + D)^{1/2}} \frac{(\det' \bar{\partial})(\hat{\mathcal{S}} \otimes F_w)}{N_\delta^2} .
\end{aligned}$$

Hence

$$\frac{(\det' \bar{\partial})(\hat{\mathcal{S}}' \otimes F_{w'})}{N_{\delta'}^2} = e^{i\xi(\delta, \Lambda)} \varepsilon(\Lambda)^{-1} e^{i\pi\phi(\delta, \Lambda)} \det(C\Omega + D)^{-1/2} \frac{(\det' \bar{\partial})(\hat{\mathcal{S}} \otimes F_w)}{N_\delta^2} . \quad (8.8)$$

Combining the above with (5.13) gives the interesting “isometry” relation that

$$(\det' \bar{\partial})(\hat{\mathcal{S}}' \otimes F_{w'}) \left(\frac{h'_{\delta'}}{N_{\delta'}} \right)^2 = e^{i\xi(\delta, \Lambda)} (\det' \bar{\partial})(\hat{\mathcal{S}} \otimes F_w) \left(\frac{h_\delta}{N_\delta} \right)^2 . \quad (8.9)$$

This result can be used to determine $e^{i\xi(\delta, \Lambda)}$ for an odd semi-characteristic δ :

$$e^{i\xi(0, \Lambda)} = e^{i\xi(\delta, \Lambda)} e^{i\pi\phi(\delta, \Lambda)} . \quad (8.10)$$

Proof. Use (7.34) and (8.6) to observe

$$\begin{aligned}
\frac{(\det' \bar{\partial})(\hat{\mathcal{S}}' \otimes F_{w'})}{N_{\delta'}^2} &= \frac{1}{\pi} \frac{(\det \bar{\partial})(\hat{\mathcal{S}}')}{\vartheta(0; \Omega')} , \\
&= e^{i\xi(0, \Lambda)} \varepsilon(\Lambda)^{-1} \det(C\Omega + D)^{-1/2} \frac{1}{\pi} \frac{(\det \bar{\partial})(\hat{\mathcal{S}})}{\vartheta(0; \Omega)} , \\
&= e^{i\xi(0, \Lambda)} \varepsilon(\Lambda)^{-1} \det(C\Omega + D)^{-1/2} \frac{(\det' \bar{\partial})(\hat{\mathcal{S}} \otimes F_w)}{N_\delta^2} . \quad (8.11)
\end{aligned}$$

Comparing (8.11) and (8.8) yields the desired result (8.10). \square

8.3. Computing $\xi(0, \Lambda)$

In this discussion we consider the conformal field theory with two $(1/2, 0)$ chiral fermions. The “partition function” Z of this CFT is the determinant section of the determinant line bundle over spin moduli space. The pullback section \tilde{Z} has the property that

$$\tilde{Z}(\mathcal{S}) = \tilde{Z}(f \cdot \mathcal{S}) \quad (8.12)$$

where $\mathcal{S} \in \text{Teich}^{1/2}(\Sigma)$ and $f \in \text{Diff}(\Sigma)$.

We consider the case where we have a diffeomorphism f that represents¹² $\Lambda \in \Gamma_{1,2}$. This diffeomorphism maps $\text{Teich}^{1/2}(\Sigma, \hat{q})$ to itself because of the definition of $\Gamma_{1,2}$. Let $\hat{\mathcal{S}} \in \text{Teich}^{1/2}(\Sigma, \hat{q})$ then using (8.12) and trivialization (7.28) we conclude that

$$\det \bar{\partial}(f \cdot \hat{\mathcal{S}}) \exp \left(- \int_{\hat{\mathcal{S}}_0}^{f \cdot \hat{\mathcal{S}}} \tilde{\mathcal{A}} \right) \hat{\sigma}_0 = \det \bar{\partial}(\hat{\mathcal{S}}) \exp \left(- \int_{\hat{\mathcal{S}}_0}^{\hat{\mathcal{S}}} \tilde{\mathcal{A}} \right) \hat{\sigma}_0.$$

We obtain the result

$$\det \bar{\partial}(f \cdot \hat{\mathcal{S}}) = \det \bar{\partial}(\hat{\mathcal{S}}) \exp \left(\int_{\hat{\mathcal{S}}}^{f \cdot \hat{\mathcal{S}}} \tilde{\mathcal{A}} \right) \quad (8.13)$$

Because of flatness we have that

$$\exp \left(\int_{\hat{\mathcal{S}}}^{f \cdot \hat{\mathcal{S}}} \tilde{\mathcal{A}} \right) = \exp \left(\int_{\hat{\mathcal{S}}_0}^{f \cdot \hat{\mathcal{S}}_0} \tilde{\mathcal{A}} \right),$$

i.e., the parallel transport factor is independent of basepoint. Any path from $\hat{\mathcal{S}}_0$ to $f \cdot \hat{\mathcal{S}}_0$ projects to a loop in $\mathcal{M}_{1,2}(\Sigma)$, where $\mathcal{M}_{1,2}(\Sigma) = \text{Teich}(\Sigma) / \text{Diff}_{1,2}(\Sigma)$ and $\text{Diff}_{1,2}(\Sigma)$ are the diffeomorphisms representing $\Gamma_{1,2}$. We see that the exponential above is just the holonomy with respect to the flat connection \mathcal{A} pulled back to the pull back line bundle over $\mathcal{M}_{1,2}(\Sigma) \supset \mathcal{M}(\Sigma)$. We know that the flat hermitian connection gives a homomorphism $\text{hol} : \pi_1(\mathcal{M}_{1,2}(\Sigma)) \rightarrow S^1$ and by relating (8.13) with (8.5) we obtain

Theorem 8.6. *If $\Lambda \in \Gamma_{1,2}$ then*

$$e^{i\xi(0,\Lambda)} = \text{hol}(\Lambda)^{-1}. \quad (8.14)$$

The holonomy is independent of the choice of representative $f \in \text{Diff}(\Sigma)$ chosen for Λ .

Remark 8.7. Because S^1 is abelian, the set of maps of $\pi_1(\mathcal{M}_{1,2}(\Sigma)) \rightarrow S^1$ is the same as the set of maps $H_1(\mathcal{M}_{1,2}(\Sigma), \mathbb{Z}) \rightarrow S^1$, *i.e.*, the character group of $H_1(\mathcal{M}_{1,2}(\Sigma), \mathbb{Z})$ which equals $H^1(\mathcal{M}_{1,2}(\Sigma), \mathbb{R}) / H^1(\mathcal{M}_{1,2}(\Sigma), \mathbb{Z})$, a torus T . Therefore our map hol is a homomorphism from $\Gamma_{1,2}$ to the torus T . Our theorem states that $e^{i\xi(0,\Lambda)} = \text{hol}(\Lambda)^{-1} \in T$. We will denote $e^{i\xi(0,\Lambda)}$ by $t_\Lambda \in T$. We can also interpret $e^{in\xi(0,\Lambda)}$ by defining $\text{hol}^n : \Gamma_{1,2} \rightarrow T$ by $\text{hol}^n(\Lambda) = (\text{hol}(\Lambda))^n$ so that $t_\Lambda^n = e^{in\xi(0,\Lambda)} = (\text{hol}^n(\Lambda))^{-1} \in T$.

Remark 8.8. The original proposal relating global anomalies to holonomy is due to Witten [20].

¹²We have learned from Igusa [19] and Mumford [8] that using $\Gamma_{1,2}$ simplifies life.

8.4. Main Results

As we noted in the introduction, the purpose of this paper is to extend the results of BEG, where we had a theory over Teichmuller space, to a theory over moduli space.

We will denote the dependence of Z_{sc} on a string manifold M and a spin structure \mathcal{S} by $Z_{\text{sc}}(M, \mathcal{S})$.

Remark 8.9. $Z_{\text{sc}}(M, \mathcal{S})$ is a function on $\text{Teich}^{1/2}(\Sigma)$ and the map $M \mapsto Z_{\text{sc}}(M, \mathcal{S})$ is a homomorphism from string cobordism MString^* to functions on $\text{Teich}^{1/2}(\Sigma)$, i.e., a genus.

It turns out it is useful to define a function Φ by

$$Z_{\text{sc}}(M, \mathcal{S}) = \left(\frac{(\det \bar{\partial})(\hat{\mathcal{S}})}{\vartheta(0; \Omega)} \right)^{2n} \Phi(M, \mathcal{S}). \quad (8.15)$$

The map $M \mapsto \Phi(M, \mathcal{S})$ is also a genus from MString^* to $\text{Teich}^{1/2}(\Sigma)$.

Formula (4.2) for Z_{sc} contained $\det' \partial$ and not $\det' \bar{\partial}$, therefore we have to take the complex conjugate. Using the notation of (4.2) we find the transformation law

$$\left(\frac{\det' \partial_{\delta'}}{\bar{N}_{\delta'}^2} \right)_{g'} = \varepsilon(\Lambda) e^{-i\xi(0, \Lambda)} \det(C\bar{\Omega} + D)^{-1/2} \left(\frac{\det' \partial_{\delta}}{\bar{N}_{\delta}^2} \right)_g. \quad (8.16)$$

A drawback of Theorem 5.1 is that the integral is a combination of objects that vary holomorphically and objects that vary anti-holomorphically with respect to Teichmuller space. For example Ω varies holomorphically while \bar{h}_{δ}^2 varies anti-holomorphically. Next we try to group terms together in such a way as to try to be as holomorphic as possible. In particular we remind the reader that our path integral viewpoint acts as a guide to our final result. We use the Belavin-Knizhnik theorem to formally factorize the determinants and also we invoke (7.34). With this in mind we make two substitutions in (4.2):

$$\left(\frac{\text{vol } \Sigma \det H}{\det_{1\perp} \Delta_0} \right)^{1/2} \longrightarrow \frac{1}{2} \frac{(\det \bar{\partial})(\hat{\mathcal{S}})}{\vartheta(0; \Omega)} \frac{(\det \partial)(\hat{\mathcal{S}})}{\overline{\vartheta(0; \Omega)}}, \quad (8.17)$$

$$\frac{\det' \partial_{\delta}}{\bar{N}_{\delta}^2} \longrightarrow \frac{1}{\pi} \frac{(\det \partial)(\hat{\mathcal{S}})}{\overline{\vartheta(0; \Omega)}}. \quad (8.18)$$

The expression for Φ now becomes

$$\Phi(M, \mathcal{S}) = \frac{1}{(4\pi)^n} \frac{1}{(\det H)^n} \left(\frac{(\det \partial)(\hat{\mathcal{S}})}{\overline{\vartheta(0; \Omega)}} \right)^{3n} \int_M \prod_{r=1}^n \frac{\mathfrak{z}_{\kappa}(x_r z(\bar{h}_{\delta}^2); \Omega)}{\vartheta[\kappa](x_r z(\bar{h}_{\delta}^2); \Omega)}. \quad (8.19)$$

Using (5.15), (8.6) and (5.18) we obtain

Theorem 8.10. *Let M be a string manifold with $\dim M = 2n$; if $\Lambda \in \text{Sp}(2g, \mathbb{Z})$ is represented by $f \in \text{Diff}(\Sigma)$, then under the action of f we have*

$$\Phi' = \varepsilon(\Lambda)^{2n} e^{-3ni\xi(0, \Lambda)} e^{in\pi\phi(\delta, \Lambda)} \det(C\Omega + D)^n \Phi, \quad (8.20a)$$

$$= \varepsilon(\Lambda)^{2n} e^{-3ni\xi(\delta, \Lambda)} e^{-2in\pi\phi(\delta, \Lambda)} \det(C\Omega + D)^n \Phi. \quad (8.20b)$$

Corollary 8.11. *Let M be a string manifold with $\dim M = 2n$; if $\Lambda \in \mathrm{Sp}(2g, \mathbb{Z})$ is represented by $f \in \mathrm{Diff}(\Sigma)$, then under the action of f we have*

$$Z'_{sc} = e^{-in\xi(\delta, \Lambda)} Z_{sc}, \quad (8.21a)$$

$$= e^{-in\xi(0, \Lambda)} e^{in\pi\phi(\delta, \Lambda)} Z_{sc}. \quad (8.21b)$$

If $\Lambda \in \Gamma_{1,2}$ then for a semi-characteristic $[\frac{a}{b}]$ we have that $\phi(a, b, \Lambda) = \frac{1}{2} \bmod \mathbb{Z}$. If $\dim M = 2n = 4k$ then $e^{-2in\pi\phi(\delta, \Lambda)} = 1$, and the spin structure $\hat{\mathcal{S}}$ maps onto the preferred spin structure $\hat{\mathcal{S}}'$. According to Igusa [19, p. 181], $\varepsilon(\Lambda)^2$ is a character of $\Gamma_{1,2}$. Note that $\varepsilon(\Lambda)^{2n} = \varepsilon(\Lambda)^{4k} = \pm 1$.

Theorem 8.12. *Let M be a string manifold with $\dim M = 2n = 4k$; if $\Lambda \in \Gamma_{1,2} \subset \mathrm{Sp}(2g, \mathbb{Z})$ is represented by $f \in \mathrm{Diff}(\Sigma)$, then under the action of f we have*

$$\Phi' = \varepsilon(\Lambda)^{2n} e^{-3ni\xi(0, \Lambda)} e^{in\pi\phi(\delta, \Lambda)} \det(C\Omega + D)^n \Phi, \quad (8.22a)$$

$$= \varepsilon(\Lambda)^{2n} e^{-3ni\xi(\delta, \Lambda)} \det(C\Omega + D)^n \Phi. \quad (8.22b)$$

Note that $\varepsilon(\Lambda)^{2n} = \pm 1$ and $e^{in\pi\phi(\delta, \Lambda)} = \pm 1$.

If $\dim M = 2n = 8l$ then $e^{in\pi\phi(\delta, \Lambda)} = 1$, $e^{in\pi\xi(\delta, \Lambda)} = e^{in\pi\xi(0, \Lambda)}$, and $\varepsilon(\Lambda)^{2n} = 1$; and we find

Theorem 8.13. *Let M be a string manifold with $\dim M = 2n = 8l$; if $\Lambda \in \Gamma_{1,2} \subset \mathrm{Sp}(2g, \mathbb{Z})$ is represented by $f \in \mathrm{Diff}(\Sigma)$, then under the action of f we have*

$$\Phi' = e^{-3ni\xi(0, \Lambda)} \det(C\Omega + D)^n \Phi, \quad (8.23)$$

$$Z'_{sc} = e^{-in\xi(0, \Lambda)} Z_{sc}. \quad (8.24)$$

The transformation laws above do not depend on the choice of spin structure δ .

Remark 8.14. We interpret (8.24) as saying that Z_{sc} is a modular function with multiplier t_Λ^{-n} ; see Remark 8.7 for notation.

Remark 8.15. We interpret (8.23) as saying that Φ is a modular form with weight n and with multiplier t_Λ^{-3n} .

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Appendix A. Isomorphism between $\Lambda^{1/2,0}(\Sigma)$ and $\Lambda^{1/2,1}(\Sigma)$

On Σ we have local complex coordinates $z = x + iy$. In isothermal coordinates the metric is $ds^2 = g_{z\bar{z}}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$. The volume element is $\sqrt{\det g} \, dx \wedge dy$. We note that $d\bar{z} \wedge dz = 2i \, dx \wedge dy$. If ω_i are the standard abelian differentials then $\int_{\Sigma} \bar{\omega}_i \wedge \omega_j = (\Omega - \bar{\Omega})_{ij}$.

If K is the canonical bundle of Σ and if $\xi(dz)^n$ and $\eta(dz)^n$ are section of K^n then the inner product is defined by

$$\langle \xi(dz)^n, \eta(dz)^n \rangle = \int_{\Sigma} (g^{\bar{z}z})^n \bar{\xi} \eta \sqrt{\det g} \, dx \wedge dy. \quad (\text{A.1})$$

The explicit conjugate linear isomorphism between $\Lambda^{1/2,1}(\Sigma)$ and $\Lambda^{1/2,0}(\Sigma)$ is constructed as follows. Let h be a section of $\Lambda^{1/2,0}(\Sigma)$ then the corresponding section ψ of $\Lambda^{1/2,1}(\Sigma)$ is given by

$$\psi = (g_{z\bar{z}} \, dz \otimes d\bar{z})^{1/2} \bar{h}. \quad (\text{A.2})$$

Appendix B. Identifying a Holomorphic Cross Section of \mathcal{L} with a ϑ Function

Our purpose is to identify a holomorphic cross section (unique up to scale) of a determinant line bundle with a theta function. We also set the notation for the algebraic geometry we need and provide an introduction for non-experts.

In this appendix we make the identification using facts about Riemann surfaces well known to algebraic geometers.

Let $J_r(\Sigma)$ denote the set of holomorphic line bundles over Σ with c_1 equal to r . Of particular interest to us is $J_0(\Sigma)$ (our $J(\Sigma)$), the set of flat line bundles, which we can identify with $\widetilde{\pi_1(\Sigma)} = \{\chi : \pi_1(\Sigma) \rightarrow S^1\} \simeq \widetilde{H_1(\Sigma, \mathbb{Z})} \simeq H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$. The last isomorphism can be described in terms of (real) closed 1-forms ω : let $\chi_{\omega}(\gamma) = \exp\left(2\pi i \int_{\gamma} \omega\right)$ for γ a closed path starting at $P_0 \in \Sigma$. Clearly χ_{ω} is a homomorphism $\pi_1(\Sigma) \rightarrow S^1$ and depends only on the cohomology class of ω . It is easy to see that $\omega \mapsto \chi_{\omega}$ induces an isomorphism $H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}) \rightarrow \widetilde{\pi_1(\Sigma)}$. Here we take $\pi_1(\Sigma)$ as the closed (piecewise smooth) paths starting at P_0 with equivalence relationship given by homotopy.

When the real surface Σ has a complex structure, then $H^1(\Sigma, \mathbb{C}) = H^1(\Sigma, \mathbb{R}) \otimes \mathbb{C} \simeq H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma)$. Taking the real part induces an isomorphism of $H^{0,1}(\Sigma)$ (or $\widetilde{H^{1,0}(\Sigma)}$) with $H^1(\Sigma, \mathbb{R})$. Let $\text{re} : H^{0,1}(\Sigma) \rightarrow H^1(\Sigma, \mathbb{R})$ be this isomorphism, and let $\widetilde{H^{0,1}(\Sigma)} = (\text{re})^{-1} H^1(\Sigma, \mathbb{Z})$ so that $\widetilde{H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})} \simeq \widetilde{H^{0,1}(\Sigma)}/\widetilde{H^{0,1}(\Sigma)}$. Since $H^{0,1}(\Sigma)$ is a complex vector space, $\widetilde{H^{0,1}(\Sigma)}/\widetilde{H^{0,1}(\Sigma)}$ is a complex torus $J(\Sigma)$, the jacobian of Σ . Our chain of arguments demonstrates that the jacobian $J_0(\Sigma)$ is isomorphic to $\widetilde{H_1(\Sigma, \mathbb{Z})}$, the character group of $H_1(\Sigma, \mathbb{Z})$. Specifically, if $\mu \in H^{0,1}(\Sigma)$, let

$$\chi_{\mu}(\gamma) = e^{2\pi i \int_{\gamma} (\mu + \bar{\mu})/2}, \quad (\text{B.1})$$

with γ a loop with basepoint P_0 , then $\mu \rightarrow \chi_\mu$ induces the isomorphism of $H^{0,1}(\Sigma)/L_\Omega = J(\Sigma)$ with $\widehat{H_1(\Sigma, \mathbb{Z})}$. $L_\Omega = \widehat{H^{0,1}(\Sigma)}$ will be described differently in the paragraph below. The covering space of $J(\Sigma) = J_0(\Sigma)$ is $H^{0,1}(\Sigma)$.

We chose a standard basis $(\bar{\omega}_1, \dots, \bar{\omega}_g)$ of $H^{0,1}(\Sigma)$ obtained from a choice of symplectic basis $(\mathbf{a}_j, \mathbf{b}_k)$ of $H_1(\Sigma, \mathbb{R})$ so that $\int_{\mathbf{a}_i} \omega_j = \delta_{ij}$. We remind the reader that the Riemann period matrix $\Omega_{ij} = \int_{\mathbf{b}_i} \omega_j$ with imaginary part $(\Omega_{ij} - \bar{\Omega}_{ij})/2i$ that is positive definite and in fact equal to $\langle \omega_i, \omega_j \rangle$. Then $\omega_i = \sum \alpha_i + \Omega_{ij} \beta_j$ where α and β are the harmonic representatives dual to the \mathbf{a} and \mathbf{b} cycles. We write a point in $H^{0,1}(\Sigma)$ as $(u_j + iv_j)\bar{\omega}_j$ where u and v are real. One can easily verify that $\widehat{H^{0,1}(\Sigma)}$ is represented by $2 \sum (m + \Omega n)_j (\Omega - \bar{\Omega})_{jk}^{-1} \bar{\omega}_k$ where $m \in \mathbb{Z}^g$ and $n \in \mathbb{Z}^g$ ¹³.

Multiplication m_L by a line bundle $L \in J_r(\Sigma)$ gives an isomorphism $m_L : J_0(\Sigma) \rightarrow J_r(\Sigma)$. In particular a spin structure \sqrt{K} , where K is the canonical bundle of Σ , gives $m_{\sqrt{K}} : J_0(\Sigma) \rightarrow J_{g-1}(\Sigma)$ with g the genus of Σ . Similarly for $P_0 \in \Sigma$ let L_{P_0} be the line bundle with divisor P_0 so that $L_{P_0} \in J_1(\Sigma)$. Then $m_{L_{P_0}} : J_0(\Sigma) \rightarrow J_r(\Sigma)$ is an isomorphism. The complex structure on J_r is chosen such that m_L is holomorphic.

One can construct a Poincaré line bundle Q_r over $J_r(\Sigma) \times \Sigma$ whose restriction to each fiber $\{L\} \times \Sigma$ is the line $L \in J_r(\Sigma)$. The holomorphic line bundle Q_r over $J_r(\Sigma) \times \Sigma$ is determined only up to a line bundle on $J_r(\Sigma)$ pulled up to $J_r(\Sigma) \times \Sigma$. A choice of point $P_0 \in \Sigma$ determines Q_r by stipulating that $Q_r|_{J_r(\Sigma) \times \{P_0\}} \simeq 1$ on $J_r(\Sigma)$. In [BEG, Appendix B] we construct such a Q_0 explicitly. We can use $(m_{L_{P_0}})^* Q_{g-1}$ instead.

Let $\bar{\partial} \otimes I_{Q_r}$ be the family of $\bar{\partial}$ operators parametrized by Q_r . Suppose \mathcal{M} is a holomorphic line bundle on $J_r(\Sigma)$ which pulled up to $J_r(\Sigma) \times \Sigma$ is $\widetilde{\mathcal{M}}$. Suppose we have modified our choice of Poincaré line bundle Q_r by $Q_r \otimes \widetilde{\mathcal{M}}$. One can show the determinant line bundle of the family $\bar{\partial} \otimes I_{Q_r \otimes \widetilde{\mathcal{M}}}$, $\text{DET}(\bar{\partial} \otimes I_{Q_r \otimes \widetilde{\mathcal{M}}})$ is isomorphic to $\text{DET}(\bar{\partial} \otimes I_{Q_r}) \otimes \mathcal{M}^{r+1-g}$. In particular, when $r = g - 1$, $\text{DET}(\bar{\partial} \otimes I_{Q_{g-1}})$ is independent of choice of Q_{g-1} .

The choice of $r = g - 1$ is special because the index of the operator $\bar{\partial} \otimes I_L$, $L \in Q_{g-1}$, is zero. Generically the operator $\bar{\partial} \otimes I_L$ is invertible. Let $\mathcal{V} = \{L \in J_{g-1}(\Sigma) \mid \bar{\partial} \otimes I_L \text{ is not invertible}\}$. \mathcal{V} is a variety in $J_{g-1}(\Sigma)$, in fact the divisor of the line bundle $\text{DET}(\bar{\partial} \otimes I_{Q_{g-1}})$. Of course, \mathcal{V} is also $\{L \in J_{g-1}(\Sigma) \mid L \text{ has a nonzero holomorphic section}\}$. Another description of $\bar{\partial} \otimes I_{Q_{g-1}}$ is obtained by choosing a spin structure, a \sqrt{K} , which we denote by $\Lambda^{1/2,0}(\Sigma)$. $m_{\sqrt{K}}$ maps $J_0(\Sigma)$ to $J_{g-1}(\Sigma)$ and the family becomes a family over $J_0(\Sigma)$, namely $\bar{\partial} : \Lambda^{1/2,0}(\Sigma) \otimes F_\chi \rightarrow \Lambda^{1/2,1}(\Sigma) \otimes F_\chi$ with $F_\chi \in J_0(\Sigma)$.

We use $m_{L_{P_0}}^{g-1}$ to compare $\text{DET}(\bar{\partial} \otimes I_{Q_{g-1}})$ with $\text{DET}(\bar{\partial} \otimes I_{Q_0})$, the latter our line bundle \mathcal{L} over $J_0(\Sigma)$. The Grothendieck-Riemann-Roch theorem implies that

$$\left(m_{L_{P_0}}^{g-1}\right)^* (\text{DET}(\bar{\partial} \otimes I_{Q_{g-1}}))$$

¹³The standard basis $\{\omega_j\}$ of $H^{1,0}(\Sigma)$ identifies $H^{1,0}(\Sigma)$ with \mathbb{C}^g . Algebraic geometers [8, p. 143] identify $H_1(\Sigma, \mathbb{C})$ with \mathbb{C}^g using the Abel map. As a consequence the lattice $L_\Omega \subset \mathbb{C}^g$ is the dual torus to the algebraic geometers' Jacobian torus.

is isomorphic to \mathcal{L} . Although it is well known that $H^0(\mathcal{L}, \mathbb{C})$ has complex dimension one, i.e., the holomorphic sections of \mathcal{L} form a one dimensional subspace [BEG, Appendix C]. We now want to identify a properly normalized holomorphic section of \mathcal{L} with a ϑ -function.

First identify $J_0(\Sigma)$, the universal cover of $J_0(\Sigma)$, with \mathbb{C}^g using the basis $\{\bar{\omega}_j\}$, where we have defined the Riemann theta function ϑ and its divisor. Let \mathcal{L}_ϑ be the holomorphic line bundle over $J_0(\Sigma)$ whose divisor pulls up to the divisor of ϑ . We learn from Riemann surface theory that there exists a spin structure $\sqrt{K} \in J_{g-1}(\Sigma)$ such that $\mathcal{L}_\vartheta = m_{\sqrt{K}}^* \text{DET}(\bar{\partial} \otimes I_{Q_{g-1}})$. Put another way, $m_{\sqrt{K}}(\text{divisor of } \mathcal{L}_\vartheta) = \mathcal{V}$. The spin structure is the one determined by the choice of symplectic basis of cycles in $H_1(\Sigma, \mathbb{R})$. In fact, the spin structure \sqrt{K} has quarf the basic quarf \hat{q} described in Section 6.3. See also [8, pp. 162].

Putting these two facts together gives $\mathcal{L} \simeq \left(m_{\sqrt{K}}^{-1} m_{L_{P_0}^{g-1}}\right)^* \mathcal{L}_\vartheta \simeq \left(m_{K^{-1/2} L_{P_0}^{g-1}}\right) \mathcal{L}_\vartheta$. The flat line bundle $K^{-1/2} L_{P_0}^{g-1}$ lies in $J_0(\Sigma)$ and is in fact κ where κ is the Riemann constant, see [8, p. 166] or [21, p. 338]. Hence \mathcal{L} is the translate of \mathcal{L}_ϑ by κ . As a result, our $\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ is the translate of ϑ by κ ; the characteristic $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ equals κ .

Lemma Appendix B.1. $\vartheta[\kappa](0) = 0$

Proof. This is a consequence of Riemann's Theorem [8, Corollary 3.6 on p. 160] or [21, p. 338]. In Mumford's conventions¹⁴ we have that $\Theta = W_{g-1} - \kappa$. The divisor W_{g-1} is the image of $\text{Sym}^{g-1}(\Sigma)$ under the Abel map, i.e., the image in $J_0(\Sigma)$ of line bundles L of Chern class $g-1$ that have a holomorphic section¹⁵. Note that the origin $O \in J_0(\Sigma)$ is in W_{g-1} (it is the image of (P_0, \dots, P_0)). Riemann's Theorem then tells us that $-\kappa \in \Theta$ but Θ is symmetric therefore $\kappa \in \Theta$. We have that $\vartheta[\pm\kappa](0) \propto \vartheta(\pm\kappa) = \vartheta(\kappa) = 0$. \square

Note that $\vartheta[\kappa](0) = 0$ if κ is an odd characteristic. Mumford [22, p. 3.82], and Farkas and Kra [23] discuss that in hyperelliptic surfaces, κ will be even or odd depending on the genus. If κ is even then $\vartheta[\kappa](\cdot)$ has a double zero at the origin and this has implications on the number of zero modes of the Dirac operator. All genus 2 surfaces are hyperelliptic and κ is an odd characteristic.

Appendix B.1. Definition of \mathfrak{z}_κ

The technicalities in constructing the determinant line bundle are related to the jump in the dimensionality of $\ker D$ at the origin $O \in J_0(\Sigma)$:

Theorem Appendix B.2 ([BEG, Theorem B.1]). *Let $\chi \in J(\Sigma)$ be a character and let $A^{0,1}$ be the associated flat connection. If $\chi = 1$ then $\dim \ker D_0 = g$. If $\chi \neq 1$ then $\dim \ker D_A = g - 1$.*

¹⁴Mumford's vector of Riemann constants is the negative of Griffiths and Harris' vector of Riemann constants.

¹⁵The image under the map $L \rightarrow L \otimes (L_{P_0}^{g-1})^{-1}$

We can exploit this theorem to give a global construction of the determinant line bundle over $J_0(\Sigma)$. Note that $\text{index } D_A = g - 1$ thus: (1) if $\chi \neq 1$ then D_A is surjective, (2) if $\chi = 1$ then the image of D_0 is $\mathbf{1}^\perp$. This motivates us to define a modified family of operators (parametrized by flat connections) by $\tilde{D}_A : \Lambda^{1,0}(\Sigma) \rightarrow \mathbf{1}^\perp$ where we have “killed the constants” in the range. Hence, these operators have numerical index g and $\dim \text{coker } \tilde{D}_A = 0$. Consequently $\ker \tilde{D}_A = g$ and the kernel of \tilde{D} is an honest holomorphic vector bundle over $J_0(\Sigma)$. The top wedge power of this vector bundle is the determinant line bundle $\text{DET } \tilde{D} \rightarrow J_0(\Sigma)$ and it is isomorphic to $\text{DET } D \rightarrow J_0(\Sigma)$. If we fix the metric then this construction is holomorphic because the $\mathbf{1}^\perp$ does not vary as we move over $J_0(\Sigma)$.

Our formula for Z_{sc} requires a specific trivialization of the determinant line bundle $\text{DET } D \rightarrow J_0(\Sigma)$. Here we briefly review the detailed construction given in [BEG, Section 7].

In a small neighborhood U of the origin we introduce the operator $\tilde{\mathcal{D}} : \Lambda^{1,0}(\Sigma) \rightarrow \Lambda^{0,0}(\Sigma)$ defined¹⁶ by

$$\tilde{\mathcal{D}}\phi = D\phi - i \int_{\Sigma} A^{0,1} \wedge \phi \quad (\text{B.2})$$

that has the property that $\dim \ker \tilde{\mathcal{D}} = g$ on U and we construct $\text{DET } \tilde{\mathcal{D}} \rightarrow U$. Note that both $\tilde{\mathcal{D}}$ and D are operators with index equal to $g - 1$. Let $V = J_0(\Sigma) - \{O\}$ then $\dim \ker D = g - 1$ on V , and we construct $\text{DET } D \rightarrow V$. On $U \cap V$ we have that $\ker D \subset \ker \tilde{\mathcal{D}}$ and an exact sequence of vector bundles

$$0 \rightarrow \ker D \rightarrow \ker \tilde{\mathcal{D}} \rightarrow \mathcal{K} \rightarrow 0.$$

The holomorphic line bundle \mathcal{K} is used to patch $\text{DET } D$ with $\text{DET } \tilde{\mathcal{D}}$ on $U \cap V$, and gives a construction of the holomorphic determinant line bundle $\text{DET } D \rightarrow J_0(\Sigma)$.

In [BEG, Sections 7 & 8] we discussed a convenient trivialization of $\text{DET } D$ over U by trivializing $\text{DET } \tilde{\mathcal{D}}$ over U . Note that $\ker \tilde{\mathcal{D}}|_O = H^{1,0}(\Sigma)$ and as a consequence of the Hodge theorem we can choose a basis $(\varpi_1, \dots, \varpi_g)$ for $\ker \tilde{\mathcal{D}}$ on U such that $\varpi_j = \omega_j + \Upsilon_j$ where $\Upsilon_j \in H^{1,0}(\Sigma)^\perp \subset \Lambda^{1,0}(\Sigma)$. In fact [BEG, eq. (8.1)] shows that if we subject a standard basis $(\omega_1, \dots, \omega_g)$ of $H^{1,0}(\Sigma)$ to a symplectic change of basis, see eq. (5.6), then $(\varpi_1, \dots, \varpi_g)$ transforms the same way.

We know [BEG, Appendix C] that $\text{DET } D \rightarrow J_0(\Sigma)$ has a unique holomorphic section θ_κ up to scale. Lemma Appendix B.1 tells us that this section vanishes at the origin. In the $(\varpi_1, \dots, \varpi_g)$ trivialization we write this section as

$$\theta_\kappa = \frac{\mathfrak{z}_\kappa}{\varpi_1 \wedge \dots \wedge \varpi_g} \quad (\text{B.3})$$

as explained in [BEG, eq. (8.10)]. In the trivialization given by lifting the line bundle to $H^{0,1}(\Sigma)$, the universal cover of $J_0(\Sigma)$, the section is given by the theta functions $\vartheta[\kappa](\cdot)$. Therefore the ratio $\mathfrak{z}_\kappa(z)/\theta[\kappa](z)$ is just the holomorphic transition function

¹⁶Note that the image of $\tilde{\mathcal{D}}$ is contained in $\mathbf{1}^\perp$ by construction.

that takes you from one trivialization of the line bundle to the other. In [BEG, eq. (9.5)] we give a formula for this transition function:

$$e^{-\pi i \sum z(\Omega - \bar{\Omega})^{-1} z} \frac{\mathfrak{z}_\kappa(z)}{\vartheta[\kappa](z)} = (\varpi_1 \wedge \dots \wedge \varpi_g) \otimes e^{-\int_0^A (\bar{\nu} - \rho)} (\omega_1 \wedge \dots \wedge \omega_g)^{-1}. \quad (\text{B.4})$$

Note that the right hand side is holomorphic and non-vanishing as required for a transition function. Therefore the ratio on the left hand side is holomorphic in U .

Remark Appendix B.3. An important consequence of the equation above is that the right hand side of the ratio is invariant under the symplectic action.

Appendix C. Genus 1 Modular Transformation Examples

We derive the string genus using (4.2) and specialize to genus 1 using the discussion in [BEG, Remark 3]. The presentation here is more complete than [BEG, Remark 3] because we are checking our modularity transformation results for general genus by applying them to genus one. There is a single odd spin structure, the theta function is the odd theta function, and consequently

$$\lim_{z \rightarrow 0} \frac{\vartheta(z; \tau)}{z} = \vartheta'(0; \tau) = -2\pi\eta(\tau)^3.$$

We also have $\omega = dz$, $\mathfrak{z} = -\vartheta'(0; \tau)z$, and we omit the δ label on the antiholomorphic spinor since there is only a single odd spin structure $\bar{h}^2 = \overline{\vartheta'(0; \tau)} d\bar{z}$. The modular transformation $\tau' = -1/\tau$ gives $z' = z/\tau$. Some examples:

$$(\bar{h}')^2 = \overline{\vartheta'(0; \tau')} d\bar{z}' \propto \overline{\tau^{3/2} \vartheta'(0; \tau)} d\bar{z} / \bar{\tau} = \bar{\tau}^{1/2} \overline{\vartheta'(0; \tau)} d\bar{z} = \bar{\tau}^{1/2} \bar{h}^2, \quad (\text{C.1})$$

in agreement with the complex conjugate of (5.13). Next we observe that

$$z'(\bar{h}')\tau = \left(\tau'_2 \overline{\vartheta'(0; \tau')} \right) \tau \propto \tau(\tau_2/\tau\bar{\tau}) \overline{\tau^{3/2} \vartheta'(0; \tau)} = \bar{\tau}^{1/2} \left(\tau_2 \overline{\vartheta'(0; \tau)} \right) = \bar{\tau}^{1/2} z(\bar{h})$$

that concurs with (5.17). The main object of interest is

$$\int_M \prod_{r=1}^n \frac{\mathfrak{z}(x_r z(\bar{h}))}{\vartheta(x_r z(\bar{h}); \tau)} \propto \int_M \prod_{r=1}^n \frac{\vartheta'(0; \tau)}{\vartheta(x_r z(\bar{h}); \tau) / (x_r z(\bar{h}))}. \quad (\text{C.2})$$

The last expression is reminiscent of the integral in (4.2) in the sense that as $x \rightarrow 0$ the denominator approaches the numerator. The modular transformation properties are:

$$\begin{aligned} \int_M \prod_{r=1}^n \frac{\vartheta'(0; \tau')}{\vartheta(x_r z'(\bar{h}'); \tau') / (x_r z'(\bar{h}'))} &\propto \int_M \prod_{r=1}^n \frac{\tau^{3/2} \vartheta'(0; \tau)}{[\tau^{1/2} \vartheta(x_r \bar{\tau}^{1/2} z(\bar{h}); \tau)] / (x_r \bar{\tau}^{1/2} z(\bar{h})/\tau)} \\ &\propto \int_M \prod_{r=1}^n \frac{\vartheta'(0; \tau)}{\vartheta(x_r \bar{\tau}^{1/2} z(\bar{h}); \tau) / (x_r \bar{\tau}^{1/2} z(\bar{h}))} \\ &\propto \bar{\tau}^{n/2} \int_M \prod_{r=1}^n \frac{\vartheta'(0; \tau)}{\vartheta(x_r z(\bar{h}); \tau) / (x_r z(\bar{h}))} \end{aligned}$$

in agreement with (5.18). The other terms are

$$\frac{\text{vol } \Sigma \det H}{\det_{1^\perp} \Delta_0} \propto \frac{\tau_2^2}{\det_{1^\perp} \Delta_0} \propto [\eta(\tau) \overline{\eta(\tau)}]^{-2} \quad (\text{C.3})$$

$$\frac{\det' \partial_{1/2}}{N^2} \propto \frac{\overline{\tau_2 \eta(\tau)}^2}{\tau_2 \vartheta'(0; \tau)} \propto \frac{1}{\eta(\tau)}. \quad (\text{C.4})$$

Abusing notation, $\det H' \propto \tau_2' \propto (\det H)/(\tau \bar{\tau})$,

$$\left(\frac{\text{vol } \Sigma \det H}{\det_{1^\perp} \Delta_0} \right)_{\tau'} = (\tau \bar{\tau})^{-1} \left(\frac{\text{vol } \Sigma \det H}{\det_{1^\perp} \Delta_0} \right)_\tau,$$

and

$$\left(\frac{\text{vol } \Sigma}{\det_{1^\perp} \Delta_0} \right)_{\tau'} = \left(\frac{\text{vol } \Sigma}{\det_{1^\perp} \Delta_0} \right)_\tau = \frac{1}{\tau_2} [\eta(\tau) \overline{\eta(\tau)}]^{-2}. \quad (\text{C.5})$$

Note that there is a tradeoff between “holomorphic factorization” in (C.3) and “modular invariance” in (C.5). Finally we observe that

$$\left(\frac{\det' \partial_{1/2}}{N^2} \right)_{\tau'} = \bar{\tau}^{-1/2} \left(\frac{\det' \partial_{1/2}}{N^2} \right)_\tau, \quad (\text{C.6})$$

and

$$\begin{aligned} \int_M \prod_{r=1}^n \frac{\mathfrak{z}(x_r z(\bar{h}))}{\vartheta(x_r z(\bar{h}); \tau)} &\propto \int_M \prod_{r=1}^n \frac{\vartheta'(0; \tau) (x_r \tau_2 \overline{\vartheta'(0; \tau)})}{\vartheta(x_r (\tau_2 \overline{\vartheta'(0; \tau)}); \tau)} \\ &\propto \tau_2^n \vartheta'(0; \tau)^n \overline{\vartheta'(0; \tau)^n} \int_M \prod_{r=1}^n \frac{x_r}{\vartheta(x_r; \tau)} \\ &\propto \tau_2^n \eta(\tau)^{3n} \overline{\eta(\tau)^{3n}} \int_M \prod_{r=1}^n \frac{x_r}{\vartheta(x_r; \tau)}. \end{aligned}$$

The modular transformation properties become

$$\begin{aligned} (\tau_2')^n \eta(\tau')^{3n} \overline{\eta(\tau')^{3n}} \int_M \prod_{r=1}^n \frac{x_r}{\vartheta(x_r; \tau')} &\propto \left(\frac{\tau_2}{\tau \bar{\tau}} \right)^n (\tau \bar{\tau})^{3n/2} \eta(\tau)^{3n} \overline{\eta(\tau)^{3n}} \int_M \prod_{r=1}^n \frac{x_r}{\vartheta(x_r; \tau')} \\ &\propto (\tau \bar{\tau})^{n/2} \tau_2^n \eta(\tau)^{3n} \overline{\eta(\tau)^{3n}} \int_M \prod_{r=1}^n \frac{x_r}{\vartheta((x_r \tau)/\tau; -1/\tau)} \\ &\propto (\tau \bar{\tau})^{n/2} \tau_2^n \eta(\tau)^{3n} \overline{\eta(\tau)^{3n}} \int_M \prod_{r=1}^n \frac{x_r}{\tau^{1/2} \vartheta(x_r \tau; \tau)} \\ &\propto (\tau \bar{\tau})^{n/2} \tau_2^n \eta(\tau)^{3n} \overline{\eta(\tau)^{3n}} \int_M \prod_{r=1}^n \frac{x_r \tau}{\tau^{3/2} \vartheta(x_r \tau; \tau)} \\ &\propto (\tau \bar{\tau})^{n/2} \tau_2^n \eta(\tau)^{3n} \overline{\eta(\tau)^{3n}} \frac{\tau^n}{\tau^{3n/2}} \int_M \prod_{r=1}^n \frac{x_r}{\vartheta(x_r; \tau)} \\ &\propto \bar{\tau}^{n/2} \left(\tau_2^n \eta(\tau)^{3n} \overline{\eta(\tau)^{3n}} \int_M \prod_{r=1}^n \frac{x_r}{\vartheta(x_r; \tau)} \right), \end{aligned}$$

in agreement with the abstract result. In fact

$$\int_M \prod_{r=1}^n \frac{\mathfrak{z}(x_r z(\bar{h}))}{\vartheta(x_r z(\bar{h}); \tau)} \propto \tau_2^n \overline{\vartheta'(0; \tau)^n} \int_M \prod_{r=1}^n \frac{\vartheta'(0; \tau)}{\vartheta(x_r; \tau) / x_r} \quad (\text{C.7})$$

which is exactly (C.2) if we group a factor of $\tau_2 \overline{\vartheta'(0; \tau)}$ with each x_r .

The string genus is the product of three terms

$$\left(\frac{\text{vol } \Sigma}{\det_{1^\perp} \Delta_0} \right)^n \times \left(\frac{\det' \partial_{1/2}}{N^2} \right)^n \times \int_M \prod_{r=1}^n \frac{\mathfrak{z}(x_r z(\bar{h}))}{\vartheta(x_r z(\bar{h}); \tau)}.$$

Each term has different modularity and holomorphicity properties; together they conspire to give something holomorphic which changes by a phase under modular transformations. There is no clear way of subgrouping the terms pairwise; there is a tradeoff between holomorphicity *vs* modularity depending on the grouping.

Note that the string genus is

$$\frac{1}{\eta(\tau)^{2n}} \int_M \prod_{r=1}^n \frac{\vartheta'(0; \tau)}{\vartheta(x_r; \tau) / x_r}. \quad (\text{C.8})$$

This should be compared with (4.2). The fact that in $g = 1$ the theta function is a function of one variable allowed us to scale out a variety of terms, namely all the non-holomorphic ones.

We can rewrite (C.8) as $\int_M \hat{s}(M, \tau)$ where $\hat{s}(M, \tau) = \hat{a}(M, \tau) / \eta(\tau)^{\dim M/2}$ and

$$\hat{a}(M, \tau) = \prod_{j=1}^{\dim M/2} \frac{ix_j / 2\pi}{\sigma(ix_j / 2\pi, \tau)} \quad (\text{C.9})$$

and σ is the Weierstrass sigma function. $\hat{s}(M, \tau)$ is the string genus because $\int_M \hat{a}(M, \tau)$ is a modular form of weight $\dim M/2$, giving a homomorphism from string cobordism to the ring of modular forms.

The string genus is a generalization of Hirzebruch's genres [24, Chapters 1, 2 & 3] which are homomorphisms from a cobordism theory (MSO^* , MU^* , MSpin^*) to the ring of integers using the appropriate power series $Q(z)$ respectively given by (L , Todd, \hat{A}). The homomorphism being $\int_M \prod_j Q(x_j)$.

In the genus 1 case, the analog of $Q(z)$ is the function of one variable $(iz/2\pi) / \sigma(iz/2\pi, \tau)$. When the genus $g > 1$, see Section 8, we get a genus from a subring of the string cobordism ring, MString^* , to a subring of the functions on $\text{Teich}^{1/2}(\Sigma)$.

Nomenclature

$\text{Met}(\Sigma)$ The space of metrics on Σ with curvature -1

$\text{Teich}(\Sigma)$ The Teichmuller space of Σ

$\text{Teich}_{\text{odd}}^{1/2}(\Sigma)$ The odd spin Teichmuller space of Σ

\hat{q} The basic quarf

\mathcal{S} A spin structure

$\hat{\mathcal{S}}$ The spin structure associated to the basic quarf \hat{q}

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